Engineering Social Learning: Information Design of Time-Locked Sales Campaigns for Online Platforms

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Many online platforms (e.g., Amazon and Groupon) offer time-locked sales campaigns as an innovative selling mechanism, whereby third-party vendors sell their products at a fixed price for a pre-specified length of time. To alleviate customers' uncertainty and to influence their inference about a product's value, platforms often display to upcoming customers some information about previous customers' purchase decisions, which are the platform's proprietary observation. Using a dynamic Bayesian persuasion framework, we formulate and study how a platform should optimally design its dynamic information provision strategy to maximize its expected revenue. We establish an equivalent reformulation of the platform's information design problem by significantly reducing the dimensionality of the platform's message space and proprietary history. Specifically, we show that it suffices for the platform to use only three messages in disclosing information: a *neutral recommendation* that induces a customer to make her purchase decision according to her private assessment about the product; and a *positive* (resp., *negative*) *recommendation* that induces a customer to make the purchase (resp., not to make the purchase) by ignoring her private assessment. We also show that the platform's proprietary history can be represented by the *net purchase position*, a single-dimensional summary statistic that computes the difference between the cumulative purchases and non-purchases made by customers who receive the neutral recommendation from the platform. Subsequently, the platform's problem can be formulated and solved efficiently as a linear program. Further, we propose and optimize over a class of heuristic policies. The best heuristic policy, which we characterize analytically, is easy-to-implement, simple-to-prescribe, and near-optimal policy. Specifically, this heuristic policy provides only neutral recommendations to customers arriving up to a cut-off customer and provides only positive or negative recommendations to customers arriving afterwards. The recommendation is positive if and only if the net purchase position achieved right after the cut-off customer exceeds a threshold. Finally, we demonstrate that the best heuristic policy improves the platform's revenue over naïve policies commonly used in practice, such as the no-disclosure and full-disclosure policies, and captures at least 90% of the optimal revenue.

Key words: revenue management; dynamic information provision; Bayesian inference; recommendation; linear program; implementation

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1. Introduction

The recent surge of digital platform economy is bringing radical changes to how we socialize, trade, and exchange information. In contrast to the businesses operated in a traditional economy, online platforms such as Amazon, eBay and Groupon play the role of a market maker by establishing technology-based infrastructures that allow a large number of independent vendors to sell their products and services to a broad range of customers. While these platforms have limited control over tangible instruments such as prices, they can manifest their value by facilitating and controlling the information flow among market participants in real time at no cost, a point we aim to demonstrate in this paper.

To that end, we focus on one particular application innovated by these online platforms, which we refer to as a *time-locked sales campaign*. This selling mechanism allows third-party vendors to sell their products at a fixed price for a pre-specified length of time, e.g., from a few hours to a few days. As the campaign progresses, the platform displays the time remaining in the campaign. Platforms charge vendors a pre-negotiated commission, i.e., a fraction of the sales.[1] However, customers often face uncertainty about the value of products, deterring them from making the purchase. This uncertainty is particularly significant for new products, products with nuanced features, or products that cater to a niche market. The lack of physical showrooms in the online environment further exacerbates the issue. As such, platforms face the challenge to maximize the sales (i) in the presence of customers' skepticism about the products, (ii) within a limited amount of time (iii) without being able to maneuver the price. To overcome this challenge, platforms may offer some information about historical purchase decisions made by previous consumers to upcoming customers. The format and granularity of information provided vary across different platforms. Motivated by the prevalence and heterogeneity of such practice, we study how a platform should design its information provision strategy for a time-locked sales campaign.

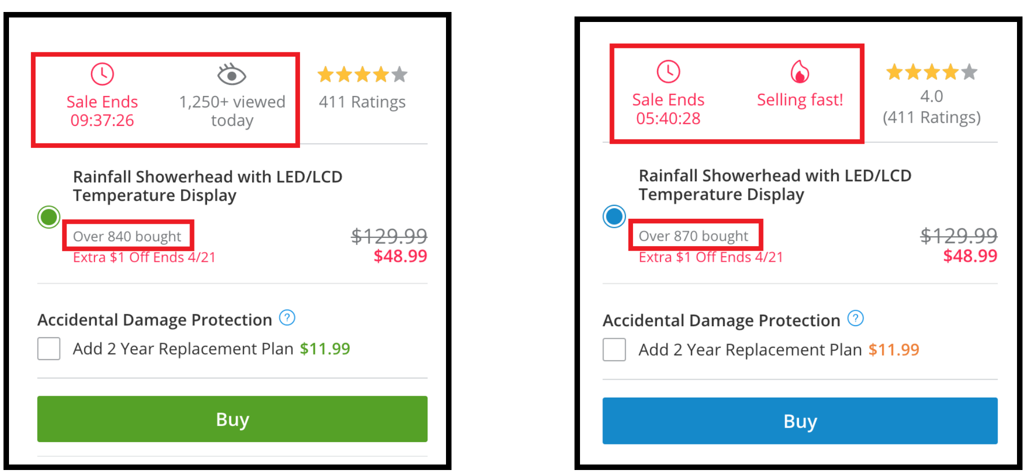
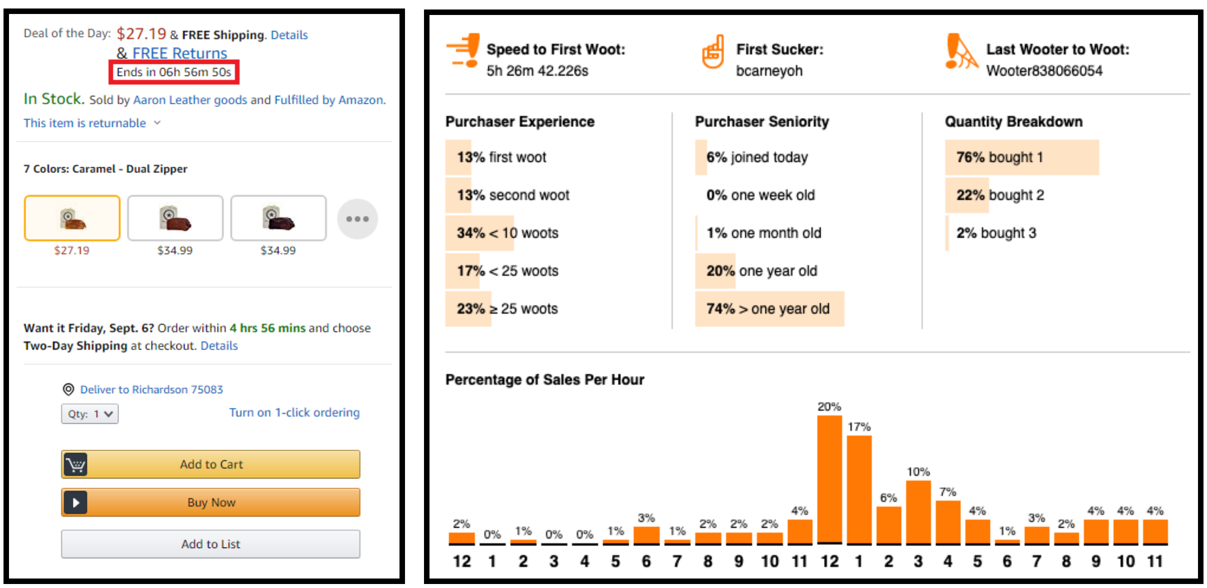


Figure 1 Key information provided for a time-locked sales campaign on Groupon.com.

[1] For example, Groupon, eBay and Amazon charge a commission rate of 8-20, 2-12 and 5-20 percent, respectively. Some platforms also charges a fixed fee per transaction, which essentially inflates the commission rate.
https://marketplace.groupon.com/support/solutions/articles/5000808521-deal-commission-rates
https://www.ebay.com/help/selling/fees-credits-invoices/selling-fees?id=4364
https://sellercentral.amazon.com/gp/help/external/200336920

The time-locked sales campaign can be best exemplified by Groupon's "Deals of the Day," as illustrated in Figure 1.[2] Customers visiting a deal's web page are presented with some descriptive information about the product (including verbal introduction, visual demonstration, existing product ratings, warranty terms, and vendor identity) as well as the price, which is set by the vendor typically at a discounted level. While these information remain constant for the entire duration of the campaign, Groupon also dynamically updates and provides customers with the time remaining to claim the deal as well as some, possibly vague, information about the up-to-date visits and sales data since the inception of the campaign. For example, in a time-locked sales campaign shown in Figure 1, Groupon displayed "1,250+ viewed today" in gray when approximately 10 hours left to claim the deal (see the left panel of Figure 1), whereas the message changed to "Selling fast!" in red 4 hours later in the same campaign (see the right panel of Figure 1).



(a) Amazon.com (b) Woot.com

Figure 2 Key information provided for a time-locked sales campaign on Amazon.com and Woot.com.

Other platforms adopt different strategies for their time-locked sales campaigns and vary in the granularity of information provided. For example, Amazon's "Deal of the Day" only demonstrates how much time left to claim the deal, as shown in Figure 2(a).[3] On the other hand, Woot, a daily deals website owned by Amazon, discloses the full time-series sales data as illustrated by Figure 2(b). As such, the information provided by Groupon is more granular than that by Amazon's "Deal of the Day" but is less granular than that by Woot.

[2] The complete screenshots of the example campaign web pages in this section are documented in Appendix D.

[3] Amazon also hosts another time-locked sales campaign called "Lightening Deal," where it shows customers the percentage of inventory that has been sold. In this paper, we only consider campaigns without inventory constraint.

A fundamental premise behind the platform's strategy is the notion of *social learning*. Each customer has access to a signal somewhat indicative of the product value, based on her research or word-of-mouth information about the same or a similar product. The signal is likely to be optimistic if the product is truly of high value and pessimistic otherwise. While each customer's signal is her *private* information, her purchase decision would reflect the nature of her signal and help other customers revise their valuation of the product. However, customers cannot directly observe each other's purchase decisions, which are the platform's *proprietary* information. Thus, to influence upcoming customers' inference about the product value and subsequently their purchase likelihood, the platform can leverage its information advantage and strategically provide some information about the previous customers' purchase decisions. In essence, the platform can engineer customers' social learning process through the design of its information provision strategy.

In this paper, we model an online time-locked sales campaign visited by sequentially arriving customers, and capture the customers' social learning process using Bikhchandani et al.'s (1992) framework (in Section 3). The platform has proprietary observation of each customer's purchase decision, which is not directly observable to other customers. Using a dynamic Bayesian persuasion game framework (e.g., Kremer et al. 2014), the platform's problem is to maximize its revenue by designing an information provision strategy that dynamically displays a message to an upcoming customer based on its proprietary information of previous customers' purchase decisions.

The platform's problem stated above is very general, since the platform can use any specific format of messages (e.g., summary statistics of the purchase history) and can dynamically adjust the information provided to the customers during the campaign. As a key methodological contribution, we establish an equivalent reformulation of the platform's problem by significantly reducing the dimension of its message space and simplifying the representation of its proprietary purchase history (in Section 4). Specifically, we show that the platform can search for its optimal information provision strategy within the class of *recommendation policies* that use only three messages: a *neutral recommendation* that induces the customer to make the purchase if and only if she receives an optimistic private signal about the product, and two *affirmative recommendations*—a *positive recommendation* and a *negative recommendation*—that induce the customer to make the purchase and not to make the purchase, respectively, regardless of her private signal. We then show that the payoff-relevant information embedded in the platform's proprietary history can be summarized through a one-dimensional summary statistic, which we termed as the platform's *net purchase position*. The net purchase position computes the difference between the cumulative numbers of purchases and non-purchases made by customers who have thus far been offered the neutral recommendation. Combining these two reductions, we are able to represent the platform's information provision strategy as a mapping from the net purchase positions to probability distributions over the three recommendations for each customer. Subsequently, the platform's problem is formulated and solved efficiently as a linear program.

For the ease of implementation, we also propose the class of *NA-partition* policies, which partition each customer according to their order of arrival, either (i) as a customer, to whom the platform can only provide the neutral recommendation regardless of the net purchase position, or (ii) as a customer, who can only be provided with an affirmative recommendation, possibly randomized between the positive and negative recommendations depending on the net purchase position. We find that the optimal NA-partition policy front-loads all neutral recommendations and then back-loads the affirmation recommendations (in Section 5). In other words, the heuristic policy provides neutral recommendations to customers arriving up to a cut-off customer and provides affirmative recommendations to customers arriving afterwards. The affirmative recommendation is positive if and only if the net purchase position exceeds a threshold. We fully characterize both the cut-off customer and the threshold. In fact, the optimal NA-partition policy resembles the information provision strategy employed by Groupon's "Deals of the Day." Our comprehensive numerical studies demonstrate that the optimal NA-partition policy brings significant revenue improvement (i.e., by 20% to 80% for a wide range of parameters) over the naïve policies, such as the no-disclosure policy used by Amazon's "Deal of the Day" and the full-disclosure policy used by Woot. In addition, this heuristic policy captures at least 90% of the revenue under the optimal recommendation policy (in Section 6).

2. Related Literature

Three streams of literature informed and inspired our research: revenue management, social learning, and information design. Here, we briefly review and discuss our contribution to each stream.

The classical revenue management research has focused predominantly on the pricing and inventory instruments as two key levers to maximizing firms' profit (see Talluri and Van Ryzin 2006, Özer and Phillips 2012, for a comprehensive survey). As a typical premise therein, customers know their private types such as valuation of the product (whereas the seller does not) and hence they have the informational advantage over the seller, who can then leverage pricing and inventory related decisions to screen customers' private valuation (e.g., Courty and Hao 2000, Gallego et al. 2008, Kuo et al. 2011, Bergemann et al. 2018, Chen et al. 2018, Chen and Shi 2019). As illustrated in the introduction, internet-based selling platforms may have reversed such informational advantage by being able to collect massive information regarding the supply and demand of products sold on a platform, posing new research challenges and opportunities as to how to best utilize those proprietary information for revenue management. Drakopoulos et al. (2018) and Lingenbrink and Iyer (2018) are among the first researchers to study the use of information provision as a novel instrument to raise revenue. They demonstrate strategic provision of (obfuscated) inventory and demand information can create availability risk and competition among buyers, who will then be induced to make early purchases. Following the same school of thoughts but in a completely different setting, we study how an online platform can provide information about its historical sales data to induce upcoming customers' purchases.

The fundamental linkage between the previous and upcoming customers is the notion of social learning (see Chamley 2004, for a comprehensive survey). In retail and service industries, such social learning takes the form of customer reviews, product ratings, and historical sales information that have become easily accessible with the advance of information technology.[4] Thus, a burgeoning literature emerges to examine various implications of social learning on revenue management theory and practice, ranging from pricing policies (Yu et al. 2015, Crapis et al. 2016, Papanastasiou and Savva 2016, Ifrach et al. 2019), control of service rate (Veeraraghavan and Debo 2009), inventory and product line strategies (Hu et al. 2015), to product design and introduction (Feldman et al. 2018, Araman and Caldentey 2016). This literature studies the setting in which both the seller and customers observe the information generated by the previous customers. This setting essentially corresponds to our full disclosure benchmark, in which the platform discloses the entire history of previous customers' observations and decisions regarding the product. Going beyond the full disclosure setting, Besbes and Scarsini (2018), Acemoglu et al. (2019) and Garg and Johari (2019) investigate whether and how fast social learning can reveal the true value of the product by using other information provision rules (e.g., summary statistics of the past reviews/ratings). In our work, we share with all the authors above the premise that social learning plays an instrumental role in resolving customers' uncertainty about the value of a product or service.[5]

Of particular interest to our research is the seminal work by Bikhchandani et al. (1992), whose social learning framework forms the building block of our model. However, different from their model, customers in our setting cannot directly observe the actions taken by previous customers. Instead, there is an information designer (i.e., the platform) who can mediate social learning by collecting and strategically providing (possibly obfuscated) historical purchase information, a framework that has recently been explored. For example, Che and Hörner (2018) study how a platform can utilize information provision to incentivize early exploration of a single product

[4] Instead of reviews and ratings, we focus on actual sales data as the platform's source of information because of two reasons. First, it is a well-known empirical fact that reviews and ratings typically suffer from various statistical biases and manipulation, and hence are not reliable in reflecting the true value of the product (e.g., Li and Hitt 2008, Hu et al. 2017, Chen et al. 2019). Second, a time-locked sales campaign is typically of short duration and the customers may even not be able to receive the product before the campaign is concluded. Hence, new reviews or ratings are not generated during the campaign.

[5] That said, other plausible explanations have been put forth to rationalize the disclosure of sales or reviews from previous customers, such as network effects (Hu et al. 2018) and signaling motivation (Subramanian and Rao 2016).

by abstracting the designer's information flow into a (two-state) exponential bandit. In a two-product setting, other researchers (e.g., Kremer et al. 2014, Papanastasiou et al. 2018, Bimpikis and Papanastasiou 2019) examine the information provision as an incentive instrument to foster experimentation of the under-explored product for a social welfare maximizing platform. Different from our setting, all these papers assume that the platform fully observes any signal generated by customers' consumption of the product and hence the customers do not possess any private information. We, on the other hand, model each customer to have a private signal that is imperfectly indicative of the product's value. This framework allows the platform's proprietary information to have a richer dynamic structure (which we capture through the notion of net purchase position), differentiating us from previous research.

From the methodological perspective, our modeling framework belongs and contributes to the fast-growing area of research on information design pioneered by Kamenica and Gentzkow (2011). Most recent research including ours aims to extend Kamenica and Gentzkow's (2011) static setting to dynamic ones for a variety of application contexts, such as Kremer et al. (2014), Renault et al. (2017), Ely (2017), Che and Hörner (2018), Alizamir et al. (2019), to name a few. As in all these papers, customers in our model are myopic and only optimize a single-period decision. However, most of these papers assume that receivers are *long-lived* in that they are able to observe messages provided by the designer from the beginning of the time horizon and hence their belief becomes a state variable in the designer's dynamic program. To capture the nature of the online platform, we instead model receivers as *short-lived* agents who can only observe the information provided to them in the periods of their arrival but not those provided to the previous customers, rendering the receivers' belief to be the entire distribution of the designer's proprietary history.

3. Model

In a time-locked sales campaign, an online *platform* (e.g., Groupon) allows a third-party *vendor* to sell a product to *customers* visiting the platform over a finite time horizon of length T. The vendor sets a constant price, denoted as p, for the entire selling horizon; thus, the platform has no control over the price. The platform profits from each sale through a pre-specified commission rate, a fraction of the selling price, which we normalize to 1 without loss of generality. The vendor receives the rest of the payments and is responsible for fulfilling the order. Hence, the vendor's revenue is proportional to the platform's revenue. To maximize its revenue, the platform's goal is to generate as many sales as possible. In this paper, we take the platform's perspective and focus on the strategic interaction between the platform and customers.

When selling an innovative or a relatively nuanced product, an online time-locked sales campaign typically faces the challenge of resolving customers' uncertainty about the product's value at the time of purchase (before consumption) due to (i) the lack of showroom for customers to experience the product and (ii) insufficient amount of time for the platform to collect and display reviews from customers, who have made purchases.[6] For simplicity, the customer enjoys a consumption utility, which we normalize to 1, from a valuable or fitting product; otherwise, the customer enjoys zero consumption utility. We thus model the customer's a priori uncertain utility from consuming the product as a binary random variable $V \in \{0,1\}$. Customer can purchase the product and receive the payoff, $V - p$; or she does not purchase the product and receive zero payoff. To rule out trivial cases, we normalize the price to $p \in (0,1)$ so that the customer's net payoff is positive if the purchased product is valuable and negative otherwise.

A priori, neither the platform nor customers know the exact value of V. At the beginning of the selling horizon ($t = 0$), the vendor publishes on the platform some general information about the product, such as verbal description, virtual demonstration or even past customers' reviews and testimony, and these information remains constant throughout the selling horizon. Based on these public information, the platform and all customers form a common prior expectation $\mathbb{E}[V] = v_0 \in [0,1]$, i.e., $V = 1$ with probability v_0 and $V = 0$ with the complimentary probability $1 - v_0$. In summary, the platform and all customers are endowed with three exogenous parameters (T, p, v_0), which define a time-locked sales campaign.

3.1. Dynamics and information

We divide the selling horizon into discrete time periods according to customers' arrival so that the time stamp also indexes the customer's order of arrival. That is, customer $t \in \{1, \ldots, T\}$, the customer arriving in time period t, is the t-th customer visiting the platform. Since the sales campaign is time-locked (i.e., the platform displays the time remaining in the campaign or equivalently the duration of the campaign that has elapsed), each customer knows her order of arrival.[7] Besides the general information about the product (represented by v_0), customers are heterogeneous and form their private assessment of the product by researching or acquiring word-of-mouth information about the same or similar product from some exogenous information channel (e.g., search engine, social media). Such private assessment can be *optimistic* or *pessimistic* in that it induces a customer to update her expectation of V above or below the prior v_0, respectively.

Formally, we represent customer t's private assessment as a binary signal S_t taking symbolic values 1 and -1, which represent an optimistic and pessimistic assessment, respectively. While the

[6] As the duration of these campaigns is short, products may not be even delivered until after the campaigns are concluded. Thus, customers will not be able to provide their reviews in time during the progress of the campaigns.

[7] Here, we assume that customers have perfect knowledge of their order of arrival. In practice, this assumption holds as many platforms also display the up-to-date count of visits. In theory, a customer can infer her order of arrival from her arrival time if customers arrive at a constant rate. In the literature, Kremer et al. (2014) make the same assumption and demonstrate that it is innocuous. Note that each customer has no control over her order of arrival, as they cannot control other customers' arrivals.

realization of S_t is customer t's private information, its distribution (conditional on the product value V) is public knowledge. Specifically, following Bikhchandani et al. (1992), we model S_t as an identical and independent binary random variable with the following distribution conditional on the value V:

$$\mathbb{P}[S_t = 1 \mid V = 1] = \mathbb{P}[S_t = -1 \mid V = 0] = q \in (1/2, 1], \tag{1}$$

where q is referred to as the signal's *precision* and measures how much indicative a customer's private signal S_t is of the value V.[8] According to the Bayes Rule, an optimistic signal improves the customer's expectation above v_0 (i.e., $\mathbb{E}[V \mid S_t = 1] = \frac{v_0 q}{v_0 q + (1-v_0)(1-q)} > v_0$) and a pessimistic signal lowers her expectation below v_0 (i.e., $\mathbb{E}[V \mid S_t = -1] = \frac{v_0(1-q)}{v_0(1-q)+(1-v_0)q} < v_0$). Notably, one could asymptotically learn the true value of V by observing a sufficient number of such private signals.

The platform has an information advantage over customers in that the platform observes the purchase decisions of *all* customers who have visited the platform as well as *all* information provided to them. Hence, the platform can leverage the better information it accrues and decides when and how to release such information so as to persuade an upcoming customer into purchase. Formally, let m_t denote the information that the platform provides to customer t. Given such information (in addition to v_0 and S_t), customer t maximizes her expected payoff by deciding whether or not to purchase. We denote customer t's decision as $a_t \in \{1, -1\}$, where $a_t = 1$ and $a_t = -1$ denote a purchase and a non-purchase, respectively. Thus, we say that customer t follows her private signal S_t to make her purchase decision if $a_t = S_t$, but it is possible for $a_t \neq S_t$, i.e., she dismisses her private signal when making her purchase decision. An arriving customer cannot directly observe the purchase decisions made by or the information provided to previous customers. Hence, customer t's purchase decision a_t and whatever information/message m_t platform provides to customer t are the platform's proprietary information. We denote the platform's *proprietary history* up to customer t as $H_t := \{(m_s, a_s) : s < t\}$ with the convention that $H_1 = \emptyset$, and the space of all possible proprietary histories as $\mathcal{H} := \{H_t : t = 1, 2, \cdots, T\}$.

3.2. Platform's information provision policy

Next, we describe how the platform generates the information m_t provided to each customer t. For generality, we allow the platform to dynamically adjust the information provided to different customers and do not impose any restriction on the format of such information (e.g., verbal or visual messages are allowed). This allows our framework to capture a wide range of applications including those illustrated in the Introduction (see Figures 1 and 2). Specifically, the platform

[8] The signal's precision depends on the quality of the information channel, through which customers acquire their signals, and hence is independent of the product value.

designs and commits to[9] an *information provision policy*, denoted as $(\sigma, \mathcal{M})$, which maps each proprietary up-to-date history $H_t \in \mathcal{H}$ to a probability distribution $\sigma(.|H_t) \in \Delta(\mathcal{M})$ over message space $\mathcal{M}$. The information provided to customer t is then a particular message $m_t \in \mathcal{M}$ drawn according to probability distribution $\sigma(.|H_t)$. For notational simplicity, we sometimes make the message space $\mathcal{M}$ implicit and abbreviate $(\sigma, \mathcal{M})$ as σ.

It is worthwhile to describe two commonly used naïve policies as polar examples of the platform's information provision policies. On one extreme, a *full-disclosure* policy, which resembles the strategy adopted by Woot, simply reveals to each upcoming customer all the previous customers' purchase decisions in its entirety; that is, $\mathcal{M} = \mathcal{H}$ and $\sigma(H_t|H_t) = 1$ for all H_t and all t. On the other extreme, a *no-disclosure* policy, which resembles the strategy adopted by Amazon's "Deal of the Day," completely conceals the platform's proprietary history by providing a constant message for all H_t and all t; that is, $\mathcal{M}$ consists of a singleton, say m_0, and $\sigma(m_0|H_t) = 1$. In this way, customers are not able to make any inference about the platform's proprietary history and hence such a policy discloses no information. We characterize the platform's revenue performance under these two naïve policies in Section 6.2.

3.3. Customers' and the platform's objectives

The platform provides message $m_t \in \mathcal{M}$ to customer t (with a prior expectation v_0) upon her arrival. Combining these information with her private signal S_t, customer t updates her belief about the product value V, and purchases the product $(a_t = 1)$ if and only if her updated expectation is not smaller than the price:[10]

$$a_t = 1 \text{ if and only if } \mathbb{E}\left[V \mid S_t, m_t, \sigma\right] \geq p, \quad \text{for all } t. \tag{2}$$

Following the convention in the literature, we assume that when the customer is indifferent between two actions, she purchases the product.

In anticipation of each customer's response provided in Equation (2), the platform's objective is to maximize its total expected sales over the entire selling horizon, by designing an information provision policy σ among all policies. That is, the platform's problem can be formulated as

$$\pi^\star := \max_{\sigma} \, p \, \mathbb{E}\left[\sum_{t=1}^{T} \mathbb{1}\left[a_t = 1\right] \middle| \, \sigma\right]. \tag{3}$$

[9] The platform's commitment means that the platform designs its information provision policy upfront at the onset of the campaign prior to the realization of its proprietary history, and abides by it as its proprietary history unravels. As articulated in the literature (see Section 2), this commitment assumption is particularly appropriate for online platforms that have reputational concerns and have to automate and computerize the implementation of their information provision for a large number of items on sale and to a wide-spread audience in real time. Further, when the sender has commitment power, she is always better off using it. Indeed, such commitment establishes the meaning of the platform's message, which would otherwise be a cheap talk and lack communicative power.

[10] Throughout the paper, all probabilities or expectations are conditional on the publicly known model primitive parameters $\{T, p, q, v_0\}$, but such dependence is kept implicit for expositional simplicity.

To summarize, we illustrate the sequence of events in Figure 3. At the campaign's onset, campaign characteristics (T, p, v_0) are defined and the platform designs an information provision policy σ according to Equation (3). Customer t arrives with her private signal S_t to the platform, receives the message m_t from the platform, and then makes her purchase decision a_t according to Equation (2). The platform updates its proprietary history $H_{t+1} = H_t \cup \{m_t, a_t\}$ and time moves on to the point when customer $t+1$ arrives at the platform. This sequence repeats until the campaign concludes with customer T.

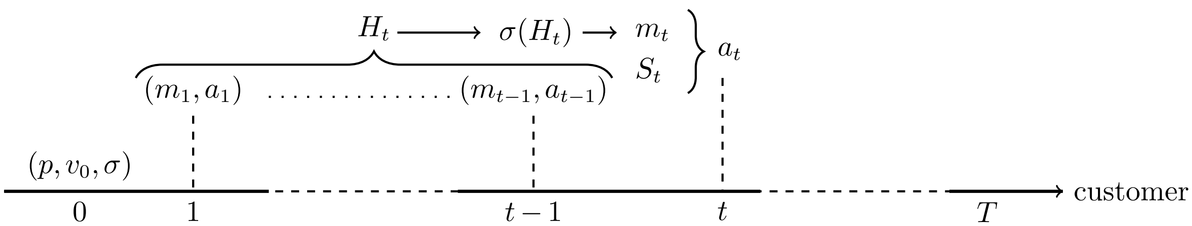


Figure 3 **Sequence of events.**

4. Reformulation and Solution of the Platform's Problem

The platform's problem expressed in Equation (3) poses significant challenges to solve due to its full generality. In this section, we establish an equivalent formulation that is mathematically solvable. The key methodological innovation to achieve this goal consists of shrinking the dimension of the message space (through a *recommendation policy*) and also simplifying the representation of the platform's proprietary history (through *net purchase position*). Along the way, we also uncover important economic insights and tradeoffs faced by the platform.

4.1. Recommendation policies and net purchase positions

To reduce the dimension of the message space $\mathcal{M}$, we first characterize customer t's purchase decision specified in Equation (2) in terms of her *interim expectation* $\mathbb{E}[V \mid m_t, \sigma]$. This interim expectation represents customer t's perceived product value based only on the information m_t the platform provides (that is, without conditioning on her private signal S_t).

PROPOSITION 1 **(Customer's purchase decision)**. *Customer t's purchase decision is*

$$a_t = \begin{cases} 1, & \text{if } \mathbb{E}[V \mid m_t, \sigma] \in [v^{**}, 1], \\ S_t, & \text{if } \mathbb{E}[V \mid m_t, \sigma] \in [v^*, v^{**}], \\ -1, & \text{if } \mathbb{E}[V \mid m_t, \sigma] \in [0, v^*], \end{cases} \tag{4}$$

where thresholds v^* and v^{**} are given by

$$v^* := \frac{p(1-q)}{p(1-q) + (1-p)q} < p < v^{**} := \frac{pq}{pq + (1-p)(1-q)}. \tag{5}$$

Proposition 1 follows from the Bayes rule. When the information provided by the platform does not adequately resolve the customer's uncertainty (i.e., when her interim expectation is between v^* and v^{**}), customer follows her private signal S_t to make the purchase decision (i.e., $a_t = S_t$). In this case, the customer purchases the product following an optimistic signal (i.e., $S_t = 1$) and does not purchase following a pessimistic signal (i.e., $S_t = -1$). In contrast, when the information provided by the platform enables the customer to form a sufficiently conclusive belief (i.e., her interim expectation is either above v^{**} or below v^*), the customer ignores her private signal S_t and makes the purchase decision according to her interim expectation. That is, she purchases the product if her interim expectation is above v^{**}, and does not purchase if it is below v^*, regardless of the value of S_t. An important consequence of Proposition 1 is the following corollary.

Corollary 1 **(Optimal policy for $v_0 \notin [v^*, v^{**})$).** *For $v_0 \notin [v^*, v^{**})$, any information provision policy is optimal for the platform. In particular, if $v_0 \geq v^{**}$, all customers make the purchase and the platform's profit is $\pi^\star = pT$; if $v_0 < v^*$, none of the customers make the purchase and the platform's profit is $\pi^\star = 0$.*

Corollary 1 shows that the platform's information design problem has a trivial solution when the common prior expectation $v_0 \notin [v^*, v^{**})$. To see this result, we note that the first customer's interim expectation is basically the prior expectation v_0, and hence, by Proposition 1, she purchases (resp., does not purchase) the product if $v_0 \geq v^{**}$(resp., $v_0 < v^{**}$) regardless of her private signal S_t. Consequently, the platform learns no new information from the first customer's purchase decision beyond prior expectation v_0. Fully aware of this fact, the second customer does not interpret the platform's second message m_2 beyond prior expectation v_0 and then behaves the same as the first customer. So do all subsequent customers, leading to Corollary 1. In other words, when customers do not have much prior uncertainty about the product value V, there is no room and need for the platform to manage its information provision to influence customers' purchase decisions. In particular, the platform can simply provide a constant message or no message at all throughout the campaign in this case. For this reason, the rest of paper will focus only on the parametric region such that $v_0 \in [v^*, v^{**})$, even though the solution methodology developed below equally applies to the complementary case.

Proposition 1 classifies the customer's purchasing strategy into three categories given by Equation (4). As such, we can equate any message provided by the platform to one of the three *incentive compatible recommendations*, symbolically denoted as $\{1, 0, -1\}$: a *positive recommendation* $m_t = 1$ that induces customer t to purchase the product regardless of her private signal ($a_t = 1$), i.e.,

$$\mathbb{E}\left[V \mid m_t = 1, \sigma\right] \geq v^{**}; \tag{IC$_1^\sigma$}$$

a *neutral recommendation* $m_t = 0$ that induces customer t to follow her private signal to make the purchase decision ($a_t = S_t$), i.e.,

$$v^* \le \mathbb{E}[V \mid m_t = 0, \sigma] \le v^{**}; \tag{IC$_0^\sigma$}$$

and a *negative recommendation* $m_t = -1$ that induces customer t not to purchase the product regardless of her private signal ($a_t = -1$), i.e.,

$$\mathbb{E}[V \mid m_t = -1, \sigma] \le v^*. \tag{IC$_{-1}^\sigma$}$$

Definition 1. We define any information provision policy $(\sigma, \mathcal{M})$ as a *recommendation policy*, if its message space consists only of the three incentive compatible recommendations $\mathcal{M} := \{1, 0, -1\}$. Since, following either a positive or negative recommendation, a customer makes a definitive purchase decision irrespective of her private signal, we refer to a positive or negative recommendation as an *affirmative recommendation*.

The three *incentive compatibility* (IC) constraints (IC$_1^\sigma$)-(IC$_{-1}^\sigma$) ensure that each recommendation induces a customer's interim expectation that is consistent with the purchase decision intended by the platform. In other words, the IC constraints establish the credibility of the platform's recommendations. Specifically, upon a positive (resp., negative) recommendation $m_t = 1$ (resp., $m_t = -1$), customer t will form her interim expectation to be above v^{**} (resp. below v^*) and subsequently make the purchase (resp., not make the purchase) regardless of her private signal S_t. Upon a neutral recommendation $m_t = 0$, customer t will form her interim expectation to be between v^* and v^{**} and subsequently follow her private signal S_t to make the purchase decision (i.e., $a_t = S_t$).

Our next proposition formally establishes that restricting the search for the optimal information provision policy within the class of recommendation policies is without loss of generality.

Proposition 2 **(Sufficiency of recommendation policy)**. *For any information provision policy, there exists a recommendation policy that induces the same purchase decisions from all customers and the same expected revenue for the platform.*

Proposition 2 drastically reduces the dimension of the message space in the platform's information provision policy. In spirit, this result is akin to the Revelation Principle in classical mechanism design problems, which allows the principal to reduce the dimension of mechanism space to that of the agent's private information. In fact, the sufficiency of recommendation policies is known for *static* Bayesian persuasion games (Kamenica and Gentzkow 2011), whereby one can establish the payoff equivalence of recommendation policies through an argument similar to the Revelation Principle. In our *dynamic* setting, we have to additionally show that the reduction to recommendation policies does not diminish the richness of information that the platform can potentially learn from customers' purchase decisions; see the proof of Proposition 2 in Appendix B.

It is worth pointing out that the particular specification of the message space as $\mathcal{M} = \{1, 0, -1\}$ is only for symbolic purpose and notational convenience. Such a recommendation policy can be implemented in practice by using messages in natural language and defining upfront the rule to generate these messages. For instance, message 1 can be framed as encouraging words such as "must buy"; message 0 can be framed as a modest suggestion such as "worth a try"; and message -1 can simply be silence.[11] Furthermore, the platform may leverage the time of providing a message as part of the message itself: verbally or visually identical messages that are provided to different customers may be interpreted as different recommendations. Namely, IC constraints (IC_1^σ)-(IC_{-1}^σ) entail different regulations on the recommendation policy for different customers.

Our next step toward obtaining the equivalent formulation for the platform's problem is to identify an efficient representation of platform's proprietary history, which is a complex object with increasing dimension as the campaign progresses. The following proposition achieves this goal.

PROPOSITION 3 **(Sufficiency of net purchase position)**. *The platform's expectation of the product value V conditional on its proprietary history H_t generated by its recommendation policy σ is*

$$\mathbb{E}\left[V \mid H_t, \sigma\right] = \frac{v_0}{v_0 + (1-v_0)(\frac{1-q}{q})^{N(H_t)}}, \tag{6}$$

where the platform's net purchase position *up to customer t is defined as*[12]

$$N(H_t) := \sum_{(m_s, a_s) \in H_t} a_s \left(1 - |m_s|\right) \in \left\{-(t-1), \cdots, 0, \cdots, t-1\right\}. \tag{7}$$

The significance of Proposition 3 lies in showing that the payoff-relevant information embedded in the platform's proprietary history is succinctly captured by a single-dimensional summary statistic of that history; that is, the net purchase position $N(H_t)$ as defined in Equation (7). Net purchase position computes the difference between the cumulative numbers of purchases and non-purchases made by customers who have thus far been offered a neutral recommendation. Note that, the net purchase position $N(H_t)$ is bounded by $-(t-1)$ and $t-1$. Recall that the platform's

[11] We remark that the firm may need to make negative recommendations (i.e., $m_t = -1$), which lead to no purchase for sure, so as to keep its positive and neutral recommendations credible. Imagine a firm that only makes positive recommendations regardless of its proprietary history. Then the positive recommendation from such a firm is completely uninformative to the customers, who in turn will not necessarily follow it (i.e., (IC_1^σ) is violated). Rather, if the firm makes negative recommendations for some realizations of its proprietary history, then a positive recommendation can carry useful information to persuade the customers to make purchase (i.e., (IC_1^σ) is satisfied). We note that the practical implementation of the negative recommendation does not need to take the form of demoting a product and can simply be *not* promoting a product, as will be illustrated at the end of Section 5.

[12] We adopt the convention that empty summation equals 0.

recommendations are incentive compatible: upon an affirmative recommendation (positive or negative), the customer finds it optimal to follow the recommendation and makes a purchase decision by dismissing her private signal. Only upon a neutral recommendation, the customer makes a purchase decision according to her private signal. In other words, the customer's purchase decision following an affirmative recommendation provides no new information for the platform to update its belief about the product value V. In contrast, the customer's purchase decision following a neutral recommendation reveals the customer's private signal, which the platform can use to revise its proprietary belief about the value of the product. In particular, customer t's purchase $a_t = 1$ (resp., no purchase $a_t = -1$) following a neutral recommendation $m_t = 0$ must imply that she has an optimistic assessment $S_t = 1$ (resp., a pessimistic assessment $S_t = -1$) about V, and hence the platform's expectation $\mathbb{E}[V \mid H_t, \sigma]$ increases (resp., decreases). Thus, the platform's expectation given in Equation (6) increases in the net purchase position.

Proposition 3 also reveals two observations. First, the platform faces the following tradeoff: A neutral recommendation enables the platform to learn about the product value V by letting the customer's purchase decision reveal her private signal S_t. In contrast, an affirmative recommendation prevents the platform from learning about the product value V from the customer's purchase decision, as it persuades the customer into a definitive purchase decision (i.e., purchase or no purchase) regardless of her private signal S_t. In other words, a neutral recommendation carries both informational and fiscal value for the platform, whereas an affirmative recommendation has only fiscal value but no informational value. Second, net purchase position essentially acts to classify the platform's proprietary histories according to the expectation they induce. That is, any two proprietary histories H_t and $H'_{t'}$, possibly of different length or even generated by different recommendation policies (say σ and σ'), would induce the same expectation as long as they yield the same net purchase positions, i.e., $\mathbb{E}[V \mid H_t, \sigma] = \mathbb{E}[V \mid H'_{t'}, \sigma']$, if $N(H_t) = N(H'_{t'})$. Therefore, the payoff-relevant information embedded in H_t is completely summarized in $N(H_t)$. For notational simplicity, we denote the platform's conditional expectation characterized by Equation (6) as

$$v_n := \mathbb{E}[V \mid N(H_t) = n, \sigma] = \frac{v_0}{v_0 + (1 - v_0)(\frac{1-q}{q})^n}, \quad \text{for any integer } n. \tag{8}$$

4.2. Equivalent reformulation of the platform's problem

Proposition 2 allows the platform to search for its optimal information provision policy within the class of recommendation policies, and Proposition 3 allows us to represent the platform's proprietary history in terms of net purchase positions. Using these two results, we can now represent a recommendation policy σ for each customer t as a mapping, denoted as r_t, from the integer space representing the set of net purchase positions n to a three-dimensional simplex representing the set of distributions over the recommendation messages $\mathcal{M} = \{1, 0, -1\}$. That is, for $n = -(T-1), \ldots, 0, \ldots, T-1$, and $t = 1, \cdots, T$,

$$r_t(n) = \left(r_t^1(n), r_t^0(n), r_t^{-1}(n)\right) \in \mathbb{R}_+^3 \text{ with } \sum_{i \in \{1,0,-1\}} r_t^i(n) = 1, \tag{R}$$

where $r_t^i(n) = \mathbb{P}[m_t = i \mid N(H_t) = n]$ is the probability that the platform offers recommendation $i \in \{1, 0, -1\}$ to customer t, given a net purchase position n.

While the platform observes its net purchase position, an arriving customer cannot. Nonetheless, given the recommendation policy $r := \{r_t : t = 1, \cdots, T\}$ committed by the platform, each customer t can form a belief (i.e., a probability distribution) about the net purchase position, which we characterize in the following proposition.

Proposition 4 **(Customer's belief of net purchase position)**. *Given a recommendation policy r, let $z_t(n) := \mathbb{P}[N(H_t) = n \mid r]$ represent customer t's belief that the platform's net purchase position is n. Then, this belief can be obtained recursively by*

$$\begin{aligned} z_t(n) &= \left(r_{t-1}^1(n) + r_{t-1}^{-1}(n)\right) z_{t-1}(n) + u_{n-1} r_{t-1}^0(n-1) z_{t-1}(n-1) \\ &\quad + (1-u_{n+1}) r_{t-1}^0(n+1) z_{t-1}(n+1), \quad \textit{for } t = 2, \cdots, T, \quad \textit{with } z_1(n) = \mathbb{1}[n=0], \end{aligned} \tag{N}$$

for all $n = -(T-1), \ldots, 0, \ldots, T-1$, where $z_t(T) = z_t(-T) = 0$ for $t = 1, \cdots, T$ and $u_n := \mathbb{P}[S_t = 1 \mid N(H_t) = n] = qv_n + (1-q)(1-v_n)$.

Given a recommendation policy r, Equation (N) shows how customer t's belief about the platform's net purchase position, $z := \{z_t : t = 1, \cdots, T\}$, evolves recursively. Note that the platform arrives at the net purchase position n from three possible net purchase positions faced by the previous customer $t-1$ (corresponding to the three terms in Equation (N)). First, the current net purchase position can remain the same as the previous one at n, if the platform provided an affirmative recommendation (positive or negative) to customer $t-1$ (because the net purchase position only accounts for purchase decisions made upon the neutral recommendation). For customer $t-1$, the net purchase position was n with probability $z_{t-1}(n)$, and the platform provided an affirmative recommendation with probability $r_{t-1}^1(n) + r_{t-1}^{-1}(n)$, leading to the first term in Equation (N). Second, the current net purchase position can increase to n from the previous one at $n-1$, if the platform offered a neutral recommendation to customer $t-1$ and the customer made a purchase. For customer $t-1$, the net purchase position was $n-1$ with probability $z_{t-1}(n-1)$, the platform provided a neutral recommendation with probability $r_{t-1}^0(n-1)$, and the customer made a purchase (i.e., received an optimistic signal $S_{t-1} = 1$) with probability u_{n-1}, leading to the second term in Equation (N). The third term can be interpreted in a similar way. In essence, the platform's net purchase position evolves according to a (non-homogeneous) random walk, whose transition probability is regulated by the platform's recommendation policy: an affirmative recommendation keeps the net purchase position unchanged, whereas a neutral recommendation splits it by one position above and below and hence disperses the customer's belief about the net purchase position.

Given the platform's expectation of the product's value in Proposition 3 and the customer's belief about the net purchase position in Proposition 4, customer t makes an inference about the product's value V from the platform's recommendation according to the Bayes rule. For example, upon receiving a positive recommendation $m_t = 1$, customer t knows that the platform is at net purchase position n with probability $z_t(n)$ and position n is pooled into message $m_t = 1$ with probability $r_t^1(n)$. Subsequently, she forms her interim expectation as $\mathbb{E}[V \mid m_t = 1, r] = \frac{\sum_n v_n z_t(n) r_t^1(n)}{\sum_n z_t(n) r_t^1(n)}$. Thus, IC constraint ($\text{IC}_1^\sigma$) on page 12 can be equivalently expressed as

$$\sum_n (v_n - v^{**}) z_t(n) r_t^1(n) \geq 0. \tag{IC_1^r}$$

By the same token, we can rewrite IC constraint (IC_0^σ) as

$$\sum_n (v_n - v^{**}) z_t(n) r_t^0(n) \leq 0, \quad \text{and} \quad \sum_n (v_n - v^{*}) z_t(n) r_t^0(n) \geq 0, \tag{IC_0^r}$$

and (IC_{-1}^σ) as

$$\sum_n (v_n - v^{*}) z_t(n) r_t^{-1}(n) \leq 0. \tag{IC_{-1}^r}$$

The IC constraints (IC_1^r)-(IC_{-1}^r) ensure that each customer find it optimal to follow the platform's recommendations to make their purchase decisions. In particular, customer t makes a purchase either upon receiving a positive recommendation, which occurs with probability $\sum_n z_t(n) r_t^1(n)$, or upon receiving a neutral recommendation and an optimistic private signal $S_t = 1$, which occurs with probability $\sum_n z_t(n) u_n r_t^0(n)$. Therefore, the platform's expected revenue from customer t can be expressed as

$$p\mathbb{E}[\mathbb{1}[a_t = 1] \mid r] = p \sum_n z_t(n) \left[r_t^1(n) + u_n r_t^0(n) \right],$$

Taken altogether, we now obtain an equivalent reformulation of the platform's problem in Equation (3) as follows:

$$\begin{aligned} \pi^\star = \max_{r,z} \quad & p \sum_{t=1}^{T} \sum_n z_t(n) \left[r_t^1(n) + u_n r_t^0(n) \right] \\ \text{subject to} \quad & (\text{R}), (\text{N}), (\text{IC}_1^r), (\text{IC}_0^r), \text{ and } (\text{IC}_{-1}^r), \end{aligned} \tag{9}$$

whose solution is denoted as $r^\star$ and $z^\star$.

4.3. Solution to the platform's problem

The platform's problem in formulation (9) is not analytically solvable primarily due to the interaction of $z_t(\cdot)$ in Equation (N) with each customer's incentive compatibility constraints (IC_1^r)-(IC_{-1}^r). Nonetheless, we are able to transform formulation (9) into a linear program and solve it efficiently, as shown by the proposition below.

PROPOSITION 5 **(Optimal information provision policy)**. *The optimal solution to formulation* (9) *is given by*

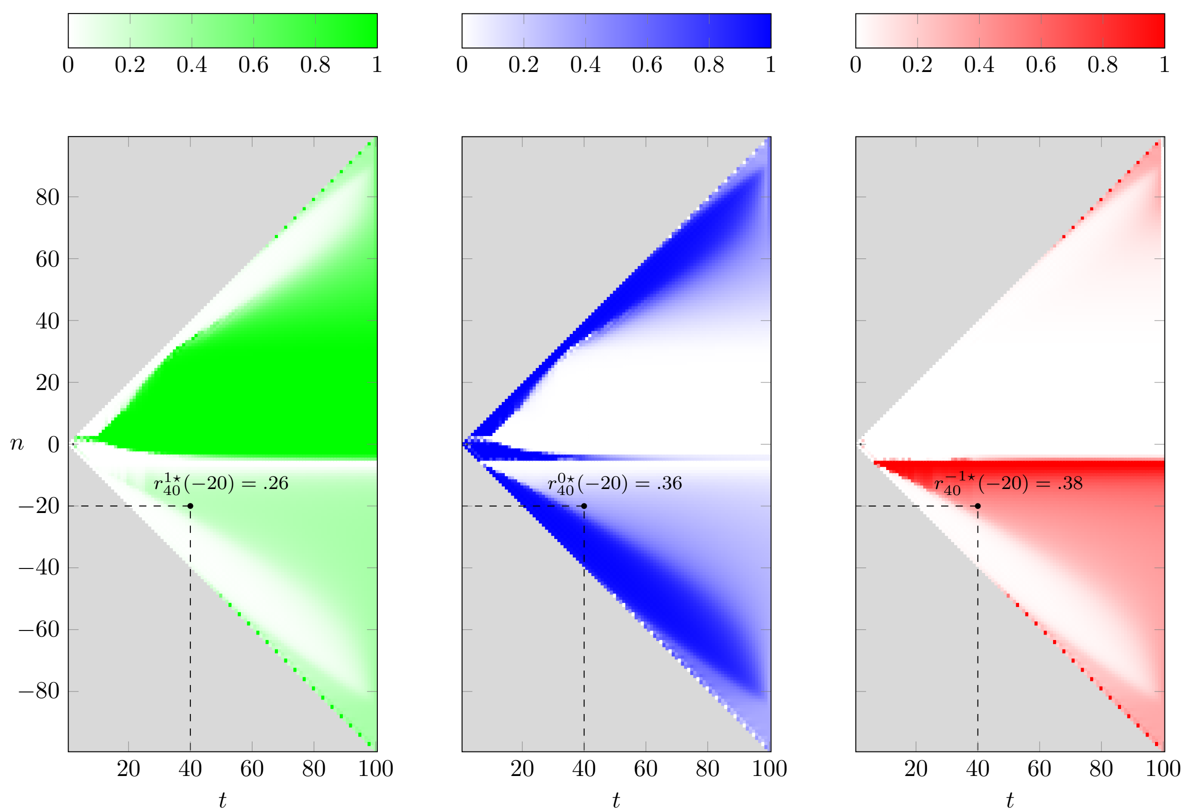
$$z_t^\star(n) = \sum_{i\in\{1,0,-1\}} y_t^{i\star}(n), \quad r_t^{i\star}(n) = \frac{y_t^{i\star}(n)}{z_t^\star(n)} \text{ for } i\in\{1,0,-1\} \text{ if } z_t^\star(n)>0, \tag{10}$$

and $r_t^\star(n)$ *being any vector satisfying Equation* (R) *if* $z_t^\star(n)=0$, *where* $y^\star$ *is the solution to the following linear program*

$$\pi^\star = \max_{y\geq 0}\; p\sum_{t=1}^{T}\sum_{n=-(T-1)}^{T-1} y_t^1(n) + u_n y_t^0(n) \tag{11}$$

$$\begin{aligned}
\text{subject to}\quad & \sum_{n=-(T-1)}^{T-1}(v_n - v^{**})y_t^1(n)\geq 0, \qquad \sum_{n=-(T-1)}^{T-1}(v_n - v^{*})y_t^{-1}(n)\leq 0,\\
& \sum_{n=-(T-1)}^{T-1}(v_n - v^{**})y_t^0(n)\leq 0, \qquad \sum_{n=-(T-1)}^{T-1}(v_n - v^{*})y_t^0(n)\geq 0, \quad \text{for } t=1,\cdots,T;\\
& \sum_{i\in\{1,0,-1\}} y_t^i(n) = y_{t-1}^1(n) + y_{t-1}^{-1}(n) + u_{n-1}y_{t-1}^0(n-1) + (1-u_{n+1})y_{t-1}^0(n+1)\\
& \text{with } y_{t-1}^0(-T) = y_{t-1}^0(T) = 0 \text{ and } \sum_{i\in\{1,0,-1\}} y_1^i(n) = \mathbb{1}[n=0],\\
& \text{for } t=2,\cdots,T, \text{ and } n=-(T-1),\cdots,T-1.
\end{aligned}$$

To arrive at the LP formulation (11), we observe that the decision variables $r_t^i(n)$ and $z_t(n)$ enter the objective and constraints of formulation (9) together as $r_t^i(n)z_t(n)$, which basically represents the joint probability that customer t faces net purchase position n *and* is recommended message $i\in\{1,0,-1\}$. Therefore, substituting $r_t^i(n)z_t(n)$ with $y_t^i(n)$ yields formulation (11). In particular, the first constraint in formulation (11) corresponds to Equation (N), and the remaining four constraints correspond to (IC_1^r)-(IC_{-1}^r). Thus, the optimal recommendation policy $r^\star$ and the induced customer's belief about the net purchase position $z^\star$ can be recovered from the optimal solution $y^\star$ of formulation (11) according to Equation (10). In particular, if it is impossible for customer t to face net purchase position n (i.e., $z_t^\star(n)=0$), then any recommendation policy satisfying Equation (R) is optimal. As a result, Proposition 5 helps further reduce the number of decision variables to $3T\times(2T-1)$ in formulation (11), as opposed to having $4T\times(2T-1)$ decision variables in formulation (9). More importantly, the problem formulation (11) can be efficiently solved by using existing LP algorithms.



(a) $r_t^{1\star}(n)$ (b) $r_t^{0\star}(n)$ (c) $r_t^{-1\star}(n)$

Figure 4 Illustration of platform's optimal recommendation policy $r^\star$ as heat maps (for $T = 100, q = .7, p = .7, v_0 = .55$ hence with $v^* = .5$, $v^{**} = .8448$ and $v_0 \in [v^*, v^{**})$). The area shaded in gray corresponds to $z_t^\star(n) = 0$, and the non-shaded area corresponds to $z_t^\star(n) > 0$. For each subfigure, the gradient of the color at location (t, n) represents the magnitude of the corresponding $r_t^{i\star}(n) \in [0, 1]$.

We illustrate the platform's optimal recommendation policy $r^\star$ for a specific example with parameters given in Figure 4. Take customer $t = 40$ for instance. The net purchase position is bounded between -39 and 39 according to Equation (7). If the net purchase position turns out to be $n = -20$, then the optimal policy provides to that customer the positive recommendation with probability $r_{40}^{1\star}(-20) = .26$, the neutral recommendation with probability $r_{40}^{0\star}(-20) = .36$, and the negative recommendation with probability $r_{40}^{-1\star}(-20) = .38$. Figure 4 represents the magnitude of the three probabilities $r_t^{i\star}(n) \in [0, 1]$ (for $i \in \{1, 0, -1\}$) as the color gradient of three heat maps, with darker color corresponding to higher probabilities, for each customer t (the horizontal coordinate) and for each possible net purchase position n faced by that customer (the vertical coordinate). The triangular area shaded in gray corresponds to the net purchase positions that are not reachable by the platform (i.e., $z_t^\star(n) = 0$). As shown by Figure 4, the platform's optimal policy involves significant level of randomization among the three recommendations $\{1, 0, -1\}$ at majority of the net purchase positions for each customer. As such, the platform's optimal policy does not permit a simple analytical characterization and hence can be challenging to prescribe, interpret, and implement in practice.

Nonetheless, the optimal policy demonstrated in Figure 4 appears to rely more on neutral recommendations for earlier customers (as shown by the darker color at majority of the reachable net purchase positions for smaller t in Figure 4(b)). For later customers, on the other hand, the optimal policy appears to rely more on affirmative recommendations with positive recommendations provided at higher net purchase positions (as shown by the darker color at majority of the reachable net purchase positions with higher n for larger t in Figure 4(a)) and negative recommendations at lower net purchase positions (as shown by the darker color at majority of the reachable net purchase positions with lower n for larger t in Figure 4(c)).

Motivated by the observations above, we ask the question: Is it possible to identify a heuristic policy that involves minimal level of randomization for the ease of implementation while still capturing the global pattern of the optimal policy to ensure close-to-optimal revenue performance? Next section answers this question.

5. NA-Partition Policy

In this section, we propose and study a class of recommendation policies, which we refer to as *NA-partition policies*. This policy partitions customers according to their order of arrival into two groups: *neutral customers* and *affirmative customers*, hence the name NA-partition. Each neutral customer is provided with only the neutral recommendation regardless of the platform's net purchase position, whereas each affirmative customer is provided with only an affirmative recommendation, which can be randomized between the positive or negative recommendation depending on the platform's net purchase position. Following the notation in Equation (R), an NA-partition policy r sets either $r_t^0(n) \equiv 1$ for all n (i.e., customer t is partitioned as a neutral customer) or $r_t^1(n) + r_t^{-1}(n) = 1 - r_t^0(n) \equiv 1$ for all n (i.e., customer t is partitioned as an affirmative customer), where $r_t^1(n)$ and $r_t^{-1}(n)$, the probabilities for an affirmative customer t to receive a positive and negative recommendation, respectively, can still depend on n. That is, $r_t^0(\cdot)$ takes values 1 or 0, and is only a function of a customer's order of arrival t but not of the net purchase position n. For this reason, we denote $r_t^0(\cdot)$ as r_t^0 for an NA-partition policy r. Clearly, the class of NA-partition policies is a subset of the general recommendation policies, which allow the platform to fully randomize among all three types of recommendation at any net purchase position (i.e., $r_t^1(n) + r_t^0(n) + r_t^{-1}(n) \equiv 1$ for all n).

The goal of the remaining section is to identify the optimal policy within the class of NA-partition policies. We recall that the main analytical complication of the platform's problem formulation (9) stems from the interaction between the belief evolution in Equation (N) and the IC constraints (IC_1^r)-(IC_{-1}^r) imposed for each customer. Since Equation (N) depends on a recommendation policy only through $r_t^0(n)$ (because $r_t^1(n)+r_t^{-1}(n)=1-r_t^0(n)$), an NA-partition policy in effect breaks such interaction by making $r_t^0(n)$ independent of n. As a result, under an NA-partition policy, we can derive the customer's belief about the net purchase position in closed form, as shown by Proposition 6 below; and hence, the revenue maximization problems for individual customers become separable, as characterized by Propositions 7 and 8. This separation reduces the platform's problem of optimizing the NA-partition policy to one that simply optimizes when and how many neutral and affirmative customers to partition, as characterized in Proposition 9.[13]

As outlined above, our first step of analysis is to characterize the customer's belief about the net purchase position, $z_t(\cdot)$, by specializing the recommendation policy r in Equation (N) to an NA-partition policy.

Proposition 6 **(Customer's belief of net purchase position under NA-partition policy).** *Under an NA-partition policy r, customer t believes that the net purchase position is $N(H_t)=n$ with probability $z_t(n)=\zeta(\ell_t^r,n)$, where $\ell_t^r:=\sum_{s<t}r_s^0$ is the total number of neutral customers the platform has partitioned before customer t and*

$$\zeta(s,n)=\binom{s}{\frac{s+n}{2}}\left[v_0 q^{\frac{s+n}{2}}(1-q)^{\frac{s-n}{2}}+(1-v_0)(1-q)^{\frac{s+n}{2}}q^{\frac{s-n}{2}}\right]\mathbb{1}\left[|n|\le s \text{ and } n\equiv s \pmod 2\right]. \tag{12}$$

Proposition 6 shows that each customer's belief about the net purchase position depends on the platform's NA-partition policy r only through a single parameter ℓ_t^r, the total number of neutral customers partitioned before customer t. By definition, the net purchase position n customer t is facing is the difference between the number of purchases and non-purchases made by neutral customers arriving before her. Thus, the total numbers of purchases and non-purchases before customer t are $\frac{\ell_t^r+n}{2}$ and $\frac{\ell_t^r-n}{2}$, respectively. Conditional on the product value $V=1$ (resp., $V=0$), a neutral customer makes the purchase if she receives an optimistic signal $S_t=1$, which occurs with probability q (resp., $1-q$). Subsequently, the number of purchases made by ℓ_t^r neutral customers follows a mixture of two binomial distributions $\mathrm{Bin}(\ell_t^r,q)$ and $\mathrm{Bin}(\ell_t^r,1-q)$ with weights $v_0=\mathbb{P}[V=1]$ and $1-v_0=\mathbb{P}[V=0]$, respectively, resulting in Equation (12). As more neutral customers are partitioned (i.e., larger ℓ_t^r), both binomial distributions becomes more dispersed, so does the customers' belief about the net purchase position (see Figure D.1 in Appendix D for an illustration).

Given a customer's belief in Proposition 6, we can examine the incentive compatibility and the platform's corresponding revenue of partitioning the customer as a neutral customer or as an affirmative customer separately. We first study the case of a neutral customer.

[13] In this section, we again focus on the parametric range such that $v_0\in[v^*,v^{**})$ to rule out the trivial cases. For $v_0\ge v^{**}$ (resp., $v_0<v^*$), the optimal policy characterized in Corollary 1 can be trivially implemented through the NA-partition policy that provides the positive (resp., negative) recommendation to all customers, i.e., $r_t^{1\star}(n)\equiv 1$ (resp., $r_t^{-1\star}(n)\equiv 1$) for all t and n.

Proposition 7 **(Neutral customer)**. *It is incentive compatible to partition any customer as a neutral customer, from whom the platform earns an expected revenue of* pu_0.

If the platform partitions a customer as a neutral customer (i.e., pools all net purchase positions uniquely to the neutral recommendation), that customer cannot make any additional inference about the platform's net purchase position and hence nor about the product value V beyond the prior expectation, i.e., $\mathbb{E}[\mathbb{E}[V \mid N(H_t)]] = v_0$. Since $v_0 \in [v^*, v^{**})$, (IC_0^r) is satisfied. Based on the prior expectation v_0, a neutral customer purchases the product with probability $u_0 = qv_0 + (1-q)(1-v_0)$, which thus generates an expected revenue of pu_0 for the platform. Notably, while it is always incentive compatible to partition all customers as neutral customers (and hence generate a revenue of pu_0T), it may not be optimal to do so as the platform can potentially increase the revenue by partitioning some customers as affirmative customers.

We now turn to the case of an affirmative customer, for whom the platform can randomize between the positive and negative recommendations $m_t \in \{1, -1\}$. In this case, the platform cannot simply pool all net purchase positions to a positive or negative recommendation as that would induce the customer to form expectation $v_0 \in [v^*, v^{**})$ about the product value V, violating the relevant IC constraints (IC_1^r) and (IC_{-1}^r). Rather, the platform needs to pool sufficiently high (resp., low) net purchase positions into the positive (resp., negative) recommendation so as to induce a posterior expectation above v^{**} (resp., below v^*). This prompts us to consider an NA-partition policy that makes the positive (resp., negative) recommendation for all the net purchase positions above (resp., below) a certain *threshold* n. We refer to such policy as *threshold affirmative policy* (n, x) and formally define it as

$$r_t^1(m) = \begin{cases} 1, \text{ for } m > n, \\ x, \text{ for } m = n, \\ 0, \text{ for } n < n, \end{cases} \quad \text{and} \quad r_t^{-1}(m) = 1 - r_t^1(m), \tag{13}$$

where $x \in [0,1]$ (resp., $1-x$) represents the probability of providing the positive (resp., negative) recommendation at position n. In fact, we show that it suffices to consider threshold affirmative policies for an affirmative customer (see Lemma C.2 in Appendix C).

Under threshold affirmative policy (n, x) specified in Equation (13) and the customer's belief characterized in Proposition 6, IC constraints (IC_1^r) and (IC_{-1}^r) can be rewritten as

$$(v_n - v^{**})\,\zeta(\ell_t^r, n)x + \sum_{m>n} (v_m - v^{**})\zeta(\ell_t^r, m) \geq 0, \quad \text{and}$$
$$(v_n - v^{*})\,\zeta(\ell_t^r, n)(1-x) + \sum_{m<n} (v_m - v^{*})\zeta(\ell_t^r, m) \leq 0, \quad \text{respetively.}$$

In particular, threshold affirmative policies $(n^{**}(\ell_t^r), x^{**}(\ell_t^r))$ and $(n^{*}(\ell_t^r), x^{*}(\ell_t^r))$ bind (IC_1^r) and (IC_{-1}^r), respectively, where

$$(n^{**}(s), x^{**}(s)) := \mathop{\arg\min}_{n\ge -s,\ x\in[0,1]} \left\{ n : (v_n - v^{**})\,\zeta(s,n)x + \sum_{m>n} (v_m - v^{**})\zeta(s,m) = 0 \right\}, \text{ and} \tag{14}$$

$$(n^{*}(s), x^{*}(s)) := \mathop{\arg\max}_{n\le s,\ x\in[0,1]} \left\{ n : (v_n - v^{*})\,\zeta(s,n)(1-x) + \sum_{m<n} (v_m - v^{*})\zeta(s,m) = 0 \right\}. \tag{15}$$

Here, multiple threshold affirmative policies may bind (IC_1^r) and (IC_{-1}^r). For instance, the threshold affirmative policy $(n,1)$ represents the same policy as the threshold affirmative policy $(n-1,0)$ according to Equation (13). As a convention, we set the threshold affirmative policy $(n^{**}(s), x^{**}(s))$ as the one with the lowest threshold and $(n^{*}(s), x^{*}(s))$ as the one with the highest threshold. Indeed, as shown by the following proposition, the threshold affirmative policies given by Equations (14) and (15) are well-defined, unique,[14] and play a critical role in determining the incentive compatibility and the optimal expected revenue of partitioning an affirmative customer.

Proposition 8 **(Affirmative customer)**. *Under an NA-partition policy r, it is incentive compatible to partition customer t as an affirmative customer if and only if $\ell_t^r \ge \tau^\circ := \min\{s\ge 0 : n^*(s) > n^{**}(s),\ \text{or}\ n^*(s) = n^{**}(s)\ \text{with}\ x^{**}(s) \ge x^*(s)\}$. If customer t is partitioned as an affirmative customer, the threshold affirmative policy $(n^{**}(\ell_t^r), x^{**}(\ell_t^r))$ is optimal and generates an expected revenue of $pF(\ell_t^r)$ for the platform, where*

$$F(s) := x^{**}(s)\zeta(s, n^{**}(s)) + \sum_{n>n^{**}(s)} \zeta(s,n), \tag{16}$$

is a non-decreasing function in s.

We note that partitioning the first customer as an affirmative customer may violate (IC_1^r) and (IC_{-1}^r), because the first customer's expectation of the product value V is given by the prior expectation $v_0 \in [v^*, v^{**})$. Thus, the platform may need to accrue sufficient evidence about the product value by partitioning enough neutral customers before it can persuade upcoming customers into sure purchase or sure no-purchase through affirmative recommendations. Indeed, the first part of Proposition 8 shows that it is incentive compatible for the platform to do so, if and only if the total number of neutral customers partitioned thus far, ℓ_t^r, is sufficiently large so that $n^*(\ell_t^r) \ge n^{**}(\ell_t^r)$ with the additional requirement $x^*(\ell_t^r) \le x^{**}(\ell_t^r)$ when the equality holds. This condition essentially ensures that threshold affirmative policy $(n^{**}(\ell_t^r), x^{**}(\ell_t^r))$ satisfies (IC_{-1}^r). To see this, we note that to relax the binding (IC_{-1}^r), one needs to use a threshold affirmative policy (n,x) with

[14] For $s=0$ and $v_0 = v^*$, Equation (14) still generates a unique solution with $n^{**}(0)=0$ and $x^{**}=0$, whereas Equation (15) generates a unique $n^*(0)=0$ but has multiple values of $x^*(0)$ as the solution, in which case we choose $x^*(0)=0$.

the threshold n lower than $n^*(\ell_t^r)$ or a probability of making positive recommendation x higher than $x^*(\ell_t^r)$ (equivalently, a probability of making negative recommendation $1-x$ lower than $1-x^*(\ell_t^r)$) if the threshold $n = n^*(\ell_t^r)$. Similarly, threshold affirmative policy $(n^*(\ell_t^r), x^*(\ell_t^r))$ satisfies (IC_1^r). In fact, we find the minimum number of neutral customers required (i.e., τ°) to be no larger than three. Hence, only a small number of neutral customers are needed before partitioning affirmative customers.

Furthermore, Proposition 8 demonstrates that the optimal way to provide affirmative recommendations to an affirmative customer is given by the threshold affirmative policy $(n^{**}(\ell_t^r), x^{**}(\ell_t^r))$. This is because this policy binds (IC_1^r) and hence maximizes the probability for that affirmative customer to receive the positive recommendation (and subsequently make the purchase). This argument establishes *myopic* optimality of this policy, i.e., the expected revenue earned from that particular affirmative customer is optimized. Nevertheless, how the platform provides affirmative recommendations to an affirmative customer does not affect the distribution of the net purchase position (see Proposition 6) and hence has no effect on the platform's revenue maximization from any other affirmative customers. (According to Proposition 7, the expected revenue earned from any neutral customer is fixed and hence is not affected either.) Therefore, this policy is also *globally* optimal when the platform aims to optimize the entire NA-partition policy for the entire campaign.

Most significantly, Proposition 8 shows that (i) once it becomes incentive compatible to partition an affirmative customer, it remains incentive compatible to do so for the rest of the campaign; and that (ii) more neutral customers partitioned prior to an affirmative customer increase the platform's optimal expected revenue earned from that affirmative customer. Intuitively, as more neutral customers are partitioned, the platform collects more information about the product's value, which only strengthens its capability of making more credible affirmative recommendations, i.e., the IC constraints become easier to satisfy. Mathematically, larger ℓ_t^r disperses the customers' belief about the platform's net purchase position; it can be shown that an incentive compatible affirmative recommendation policy for a customer facing less dispersed distribution of the net purchase position can always be "replicated" (i.e., generate the same inference about product value V and the same expected revenue) for a customer facing more dispersed distribution of the net purchase position. As such, partitioning more neutral customers early in the campaign helps improve the platform's expected revenue extracted from a subsequent affirmative customer.

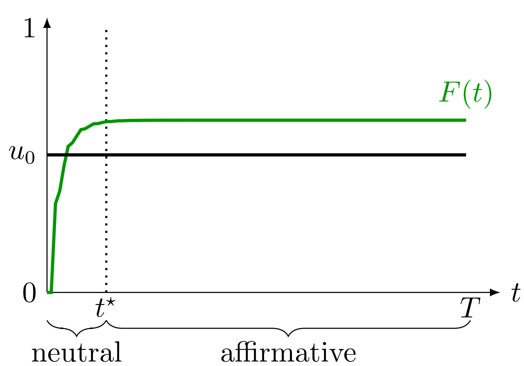
The above monotonicity property of the revenue extracted from an affirmative customer implies that the platform will be better off by partitioning the customers arriving earlier in the campaign as neutral customers and relegating all affirmative customers afterwards, resulting in the following characterization of the overall optimal NA-partition policy.

PROPOSITION 9 **(Optimal NA-partition policy)**. *Suppose $T \geq \tau^\circ - 1$.*[15] *Then the optimal NA-partition policy partitions the first $t^\star$ customers as neutral customers (i.e., $r_t^{0\star}(n) \equiv 1$ for all n and $t \leq t^\star$) and partitions all remaining customers as affirmative customers, where $t^\star$ is the solution to*

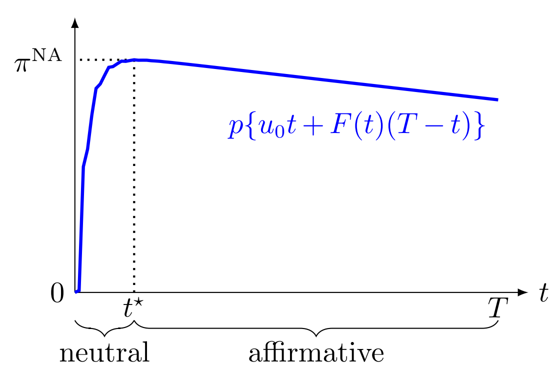
$$\pi^{\mathrm{NA}} := p \max_{t \in [\tau^\circ - 1, T]} u_0 t + F(t)(T - t). \tag{17}$$

The optimal NA-partition policy uses the same threshold affirmative policy $(n^{**}(t^\star), x^{**}(t^\star))$ for all $T - t^\star$ affirmative customers. Under the optimal NA-partition policy, the platform's total expected revenue is π^{NA}.

The optimal NA-partition policy characterized by Proposition 9 is simple to implement and entails minimal randomization: the platform partitions all customers arriving up to a *cut-off customer $t^\star$* as neutral customers irrespective of the net purchase positions they are facing (i.e., front-loads the neutral recommendation), and partitions all customers arriving afterwards as affirmative customers (i.e., front-loads the affirmative recommendation). For all the affirmative customers, the platform applies the same threshold affirmative policy $(n^{**}(t^\star), x^{**}(t^\star))$, which is the one optimized myopically for the $(t^\star + 1)$st affirmative customer. Therefore, the optimal NA-partition policy can be fully prescribed by only *three* parameters $t^\star$, $n^{**}(t^\star)$ and $x^{**}(t^\star)$. This is a significant reduction in the computational complexity compared to the LP formulation of the platform's original problem obtained in Proposition 5, which is in the magnitude of $3T \times (2T - 1)$.



(a) $F(t)$.



(b) $p\{u_0 t + F(t)(T - t)\}$.

Figure 5 **Determination of the optimal number of neutral customers $t^\star$ (for $T = 100, q = .7, p = .7, v_0 = .55$ hence with $v^* = .5$, $v^{**} = .8448$ and $v_0 \in [v^*, v^{**})$).**

Front-loading more neutral customers increases the platform's expected revenue earned from each affirmative customer (by Proposition 8), but reduces the number of affirmative customers that

[15] If $T < \tau^\circ - 1$, the only incentive compatible NA-partition policy is to partition all T customers as neutral customers. However, as discussed earlier, τ° is below three, whereas T is typically much larger in practice. Thus, this case is less relevant.

the platform can back-load due to the limited campaign length. The optimal number of neutral customers $t^\star$ characterized by Equation (17) strikes the platform's trade-off between these two effects, as illustrated by Figure 5(b). In fact, as shown in Figure 5(a), the platform should keep partitioning customer t as a neutral customer till $F(t) > u_0$, only after which the optimal expected revenue earned from an affirmative customer becomes higher than that from a neutral customer.

Implementation of optimal NA-partition policy (Best-seller mechanism): There are many approaches to implement the optimal NA-partition policy in practice. One easy implementation is what we refer to as the *best-seller* mechanism, and it works as follows:

To the first $t^\star$ customers, the platform keeps silent or provides less informative messages. For the remaining $T - t^\star$ customers, the platform *only* promotes the product ("best-seller") if the cumulative purchases made by the first $t^\star$ customers exceed $n^{**}(t^\star)/2$; otherwise, keeps doing the same as for the first $t^\star$ customers for the remaining $T - t^\star$ customers of the campaign.

In the best-seller mechanism, we leverage the customers' order of arrival to differentiate the meaning of messages before and after the cut-off customer $t^\star$: the silence or messages before the cut-off $t^\star$ should be interpreted as the neutral recommendation as it applies regardless of the platform's net purchase position, whereas those after the cut-off $t^\star$ should be interpreted as the negative recommendation as it applies only when the cumulative purchases made by the first $t^\star$ customers fail to exceed $n^{**}(t^\star)/2$. In fact, the best-seller mechanism resembles the strategy adopted by Groupon's "Deals of the Day" described in the Introduction (see Figure 1). To illustrate, Figure D.5 in Appendix D records the information provided over the course of the selling campaigns for 15 products on Groupon's "Deals of the Day."

6. Performance of NA-Partition Policy and Value of Information Design

In this section, we conduct a comprehensive numerical study to (i) evaluate the revenue performance of the optimal NA-partition policy, (ii) quantify its value against naïve policies commonly used in practice, and (iii) examine how key campaign parameters affect the optimal NA-partition policy. Our analysis leverages and complements the theoretical development of the last two sections. We also provide insights into how an online platform can effectively manage its information flow.

6.1. Performance of optimal NA-partition policy

In Proposition 5, we have shown the platform's optimal recommendation policy and expected revenue ($\pi^\star$) can be efficiently identified through a linear program. However, such an optimal policy may be difficult to implement in practice. As shown by Proposition 9, this challenge can be overcome by the optimal NA-partition policy. We now demonstrate that the revenue under the optimal NA-partition policy (π^{NA}) is close to the optimal revenue $\pi^\star$. Specifically, fixing a campaign length $T \in \{50, 100, 500\}$, we compute the optimality gap between the optimal NA-partition policy and the optimal policy, measured by the ratio $(\pi^\star - \pi^{\mathrm{NA}})/\pi^\star$, by varying the price p from 0.1 to 0.9, the prior expectation v_0 from 0.05 to 0.95, and the precision of the customer's private signal q from 0.55 to 0.95, all with 0.1 increment (see Tables D.1 and D.2 in Appendix D for details). Thus, the total number of instances for each given T is $9 \times 10 \times 5 = 450$, out of which 182 instances satisfy $v_0 \in [v^*, v^{**})$.[16] As shown by Table 1, the optimal NA-partition policy is able to garner on average 95% of the optimal revenue and more than 90% of the optimal revenue for the majority of instances. In the worst case, at least 75% of the optimal revenue can be captured by the optimal NA-partition policy.

Table 1 Summary statistics of the optimality gap between optimal NA-partition policy and optimal recommendation policy $(\pi^\star - \pi^{\mathsf{NA}})/\pi^\star$ across 182 instances of (p, q, v_0) such that $v_0 \in [v^*, v^{**})$ for $T \in \{50, 100, 500\}$.

		$T = 50$	$T = 100$	$T = 500$
Average		5.12%	4.66%	4.11%
Standard deviation		4.45%	4.51%	4.67%
Minimum		0	0	0
Maximum		24.49%	24.85%	25.13%
Number of instances with gap	$\geq 10\%$	25	22	21
	$\geq 15\%$	9	8	8
	$\geq 20\%$	3	3	2

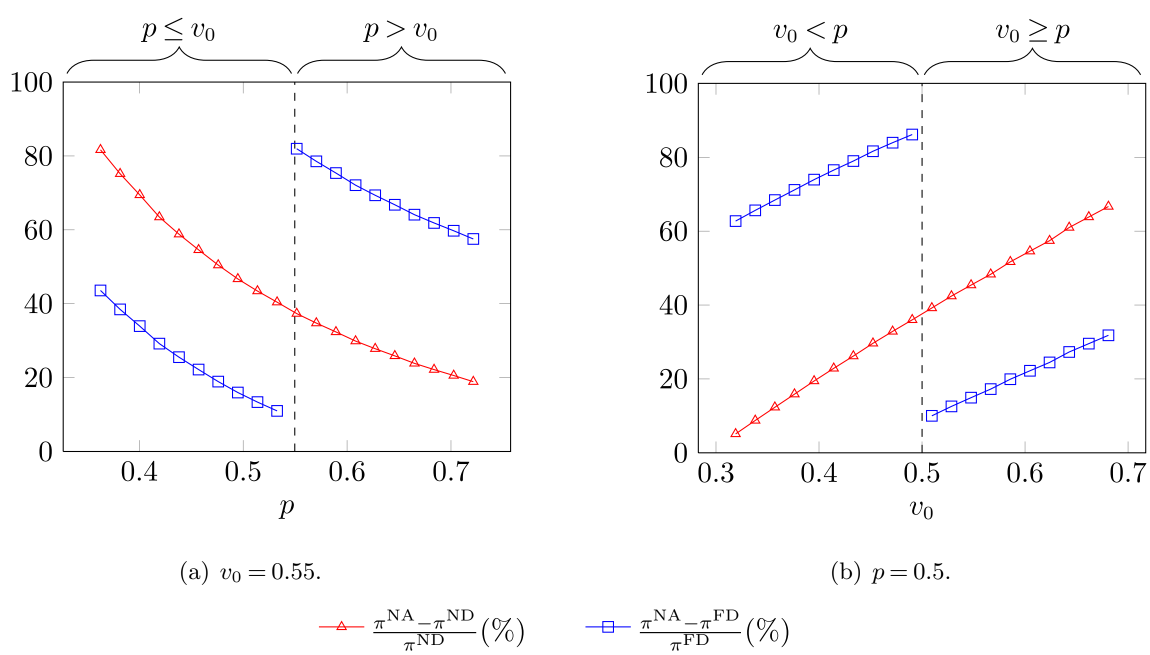
6.2. Value of information design

Two naïve information policies widely used in practice are the no-disclosure (as in the case of Amazon) and full-disclosure (as in the case of Woot) policies. They represent two extreme modes of information provision. Here we quantify how much additional revenue the platform earns under the optimal NA-partition policy identified in Proposition 9. To do so, we first characterize the platform's revenue under the two benchmark policies.

Under the no-disclosure policy, the platform withholds any information about its proprietary history from upcoming customers, who thus make their purchase decisions purely based on the prior expectation $v_0 \in [v^*, v^{**})$. By Proposition 1, therefore, all customers simply follow their private signals to make their purchase decisions, resulting in an expected revenue equal to that under an NA-partition policy that partitions all customers as a neutral customer (see Proposition 7). Thus, the platform's expected revenue under the no-disclosure policy is $\pi^{\mathrm{ND}} := p u_0 T$.

[16] For parameters such that $v_0 \notin [v^*, v^{**})$, there is no optimality gap, i.e., $\pi^{\mathrm{NA}} = \pi^\star$.

In contrast, the full-disclosure policy equalizes the information between the platform and customers, a setting explored by Bikhchandani et al. (1992). In this case, each customer observes the platform's net purchase position and bases her purchase decision on the belief identified in Proposition 3. Once a customer's posterior expectation exceeds v^{**} (resp., falls below v^*), *positive* (resp., *negative*) *information cascade* occurs: all the subsequent customers will make the purchase (resp., not make the purchase) irrespective of their private signals, and hence their purchase decisions will generate no new information about the product value. Prior to the information cascade, customers make their purchase decision according to their private signals. Proposition C.1 in Appendix C fully characterizes the platform's expected revenue π^{FD} under the full-disclosure policy. Notably π^{FD}, as a function of two main product characteristics, prior expectation v_0 and price p, is discontinuous at $v_0 = p$, as the occurrence of information cascade follows different patterns for $v_0 \geq p$ versus for $v_0 < p$.[17] The former (resp., latter) case represents a product with a promising prospect and a reasonable price (resp., a less promising and pricier product) that a customer would purchase (resp., not purchase) purely based on the publicly available information about the product without referring to her private signal nor the platform's message.



(a) $v_0 = 0.55$.

(b) $p = 0.5$.

Figure 6 Relative revenue performance of optimal NA-partition policy π^{NA} against no-disclosure policy π^{ND} and full-disclosure policy π^{FD} (for $q = .7$ and $T = 100$), plotted for the range of $v_0 \in [v^*, v^{**})$.

[17] As shown by (C.33) and (C.34) in Appendix C, for $v_0 \geq p$, the positive (resp., negative) cascade *can* occur after odd (resp., even) number of customers starting from the second (resp., third) customer, whereas for $v_0 < p$ the positive (resp., negative) cascade *can* occur after even (resp., odd) number of customers starting from the third (resp., second) customer.

To demonstrate the value of information design, we now benchmark the revenue under the optimal NA-partition policy, π^{NA}, characterized in Proposition 9 against those of the no-disclosure and full-disclosure policies.[18] The measurements we adopt are the percentage increase of π^{NA} relative to the revenue under the benchmark policies, namely $(\pi^{\mathrm{NA}} - \pi^{\mathrm{ND}})/\pi^{\mathrm{ND}}$ and $(\pi^{\mathrm{NA}} - \pi^{\mathrm{FD}})/\pi^{\mathrm{FD}}$. In Figure 6, we compute and plot these two ratios by varying the price p and the prior expectation v_0 respectively while fixing all the other parameters.

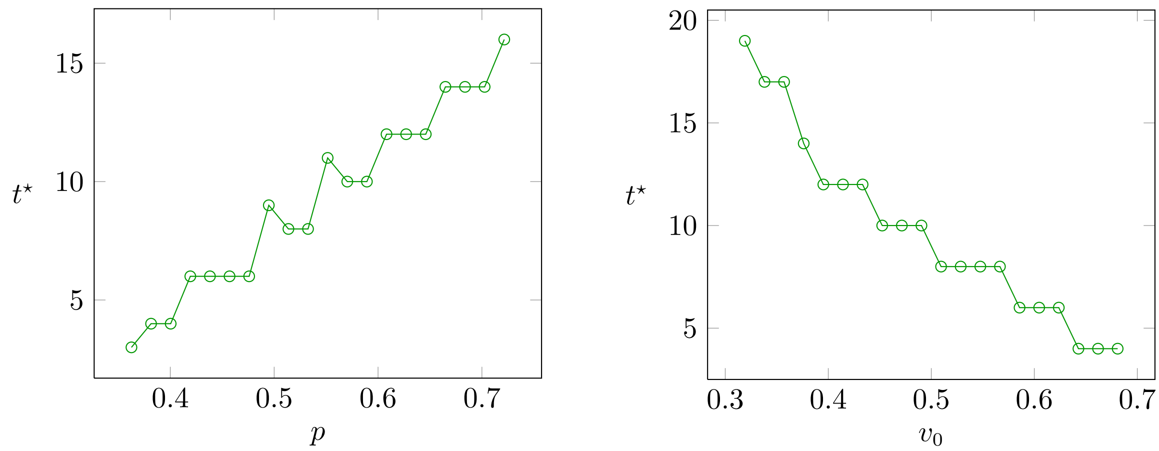
As can be seen from Figure 6, the optimal NA-partition policy always outperforms the two naïve policies and can increase the revenue under those two policies significantly (from 20% to 80% for most of the parametric instances). Specifically, relative to no disclosure, the optimal NA-partition policy manifests its value in information *provision* and this value becomes higher for products with lower prices or more promising prospects (i.e., $(\pi^{\mathrm{NA}} - \pi^{\mathrm{ND}})/\pi^{\mathrm{ND}}$ is decreasing in p and increasing in v_0, as shown by Figures 6(a) and 6(b), respectively). Relative to full disclosure, the optimal NA-partition policy manifests its value in information *obfuscation* and this value $((\pi^{\mathrm{NA}} - \pi^{\mathrm{FD}})/\pi^{\mathrm{FD}})$ demonstrates a discontinuity in the product's price p or prior prospect v_0 at $p = v_0$, as can be seen from Figures 6(a) and 6(b), respectively. Such a discontinuity emerges from the discontinuity of π^{FD} as pointed out above. Thus, Figure 6 suggests that the optimal NA-partition policy can bring a higher revenue improvement over the full-disclosure policy for $p > v_0$ than for $p \leq v_0$ (i.e., $(\pi^{\mathrm{NA}} - \pi^{\mathrm{FD}})/\pi^{\mathrm{FD}}$ is higher for $p > v_0$ than for $p \leq v_0$).

Furthermore, by comparing the two benchmark policies, Figure 6 reveals that the full-disclosure policy generates a higher revenue than the no-disclosure policy for $p \leq v_0$, as the optimal NA-partition policy yields a lower revenue improvement over the full-disclosure policy than over the no-disclosure policy (i.e., $(\pi^{\mathrm{NA}} - \pi^{\mathrm{ND}})/\pi^{\mathrm{ND}} > (\pi^{\mathrm{NA}} - \pi^{\mathrm{FD}})/\pi^{\mathrm{FD}}$ implies that $\pi^{\mathrm{FD}} > \pi^{\mathrm{ND}}$). It is the other way around for $p > v_0$. In other words, if we regard the set of all information provision policies as a continuum spectrum ranging from the no-disclosure to full-disclosure policies, then the optimal NA-partition policy seems to be closer to the full-disclosure policy (resp., the no-disclosure policy) for products with more (resp., less) promising prospects and/or lower prices (resp., higher prices).

Indeed, as indicated by Figure 7, the number of neutral customers $t^\star$ partitioned by the optimal NA-partition policy is in general increasing in p and decreasing in v_0:[19] by partitioning a larger number of neutral customers, the platform discloses less of its proprietary information and hence makes the NA-partition policy closer to the no-disclosure policy. In essence, the optimal NA-partition policy acts to optimize the occurrence of information cascade to $t^\star$ (rather than the first

[18] As already shown in section 6.1, π^{NA} is close to $\pi^\star$. Thus, similar findings can be obtained if we benchmark the optimal revenue $\pi^\star$ against the revenue under the no-disclosure and full-disclosure policies; see Figure D.6 in Appendix D for computations of $\frac{\pi^\star - \pi^{\mathrm{ND}}}{\pi^{\mathrm{ND}}}$ and $\frac{\pi^\star - \pi^{\mathrm{FD}}}{\pi^{\mathrm{FD}}}$.

[19] Local non-monotonicity in $t^\star$ is mainly caused by the discreteness in $t^\star$.



(a) $v_0 = 0.55$. (b) $p = 0.5$.

Figure 7 Optimal number of neutral customers $t^\star$, for $T = 100$ and $q = .7$.

time that the customer's posterior expectation falls outside of $[v^*, v^{**})$ under the full-disclosure policy) and maximizes the probability of a positive information cascade to $F(t^\star)$.

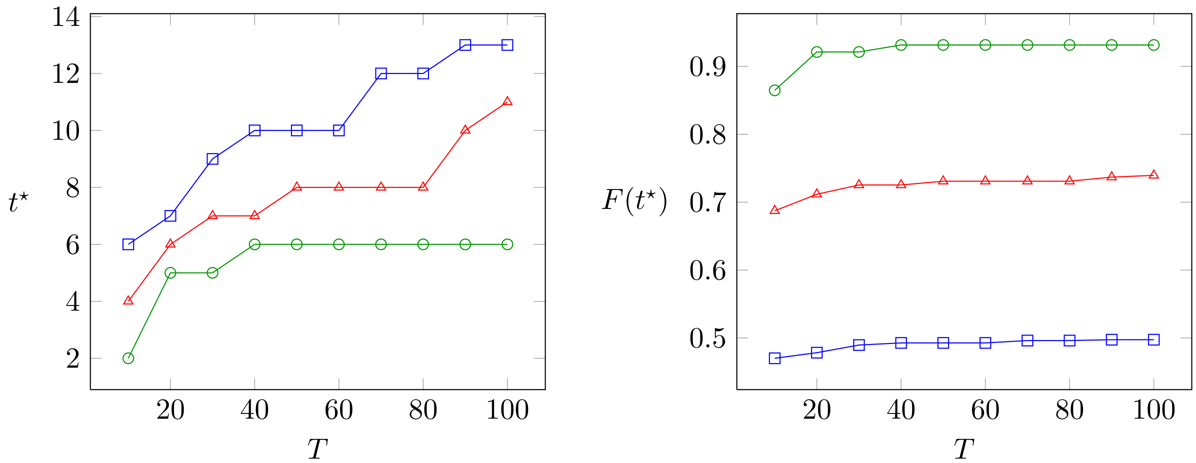
The practical implication of our observations above is evident. For products with less promising prospect (e.g., with mediocre existing ratings) or pricier products, platforms should withhold the sales history for a longer time and accrue more convincing evidence (by letting consumers make their purchase decisions based on their own private assessment) before making an affirmative recommendation. However, platforms do not need to wait too long to provide affirmative recommendations for highly rated or heavily discounted products.

6.3. Characteristics of optimal NA-partition policy

Finally, we examine the parametric dependence of the optimal NA-partition policy on the campaign length T and the product price p. Since these two parameters can potentially be adjusted by the platform and the third-party vendor, respectively, their effects are of particular interest.

Effects of campaign length T. Given price p and prior expectation v_0, Figure 8(a) shows that a longer campaign (i.e., larger T) calls for partitioning a larger number of neutral customers (i.e., larger $t^\star$). As shown by Figure 8(b), the platform in turn provides the positive recommendation (and hence induce purchases) with higher probability to the subsequent affirmative customers (i.e., larger $F(t^\star)$). Intuitively, a longer campaign allows the platform to partition more neutral customers and hence to raise its revenue extracted from each affirmative customer (see Proposition 8), while still having a sufficient number of affirmative customers for revenue extraction.

Perhaps surprisingly, Figure 8(a) indicates that the optimal NA-partition policy only partitions less than 20% of the customers as neutral customers (i.e., $t^\star/T \leq 20\%$). As indicated by Figure 8(b), such a small number of neutral customers can go a long way by significantly boosting $F(t^\star)$



(a) Optimal number of neutral customers $t^\star$.

(b) Probability of providing positive recommendation to affirmative customers $F(t^\star)$.

$p = .3,\ v_0 = .25$ $p = .55,\ v_0 = .55$ $p = .8,\ v_0 = .85$

Figure 8 Effects of campaign length T on optimal NA-partition policy (for $q = .7$).

and hence the revenue extracted from affirmative customers.[20] In fact, we recall from Figure 5(a) that $F(t)$ is increasing in t at a diminishing rate: it rises steeply for the first few neutral customers and then flattens out subsequently.

Effects of price p. Given all other campaign characteristics (T, q, v_0), we now examine how the product price p affects the platform's expected revenue under the optimal NA-partition policy. This question is of particular interest, because the third-party vendor or sometimes the platform can adjust the price p to maximize their revenues, which are proportional to each other.

We recall that price p determines the threshold value v^* and v^{**} through (5). Interestingly, Figure 9 indicates that the revenue-maximizing price should be set at either (i) the lower price limit that equates v^{**} to v_0 as illustrated by the instance of $(v_0, q) = (.65, .55)$ or (ii) the upper price limit that equates v^* to v_0 as illustrated by the instances of $(v_0, q) = (.45, .85)$ and $(v_0, q) = (.55, .7)$. In the former case (i.e., $v_0 = v^{**}$), all customers make the sure purchase regardless of their private signals, eliminating the need for any information design/provision (see Corollary 1). In contrast, the need for information design is significant in the latter case (i.e., $v_0 = v^*$): as suggested by Figure 7(a), the corresponding optimal NA-partition policy is nontrivial and calls for the largest number of

[20] When $q = .7$, the purchase probability of a neutral customer (i.e., based on prior expectation v_0) is $u_0 = 0.4, 0.52$ and 0.64 for $v_0 = 0.25, 0.55$ and 0.85, respectively. As shown by Figure 8(b), the purchase probability (i.e., the probability of making positive recommendation) of an affirmative customer increases to $F(t^\star) \approx 0.5, 0.7$, and 0.95 for these three values of v_0, corresponding to an increase over u_0 by $F(t^\star)/u_0 - 1 \approx$25%, 35% and 48%, respectively.

neutral customers (relative to under a non-optimized price).[21] Our numerical experiment suggests that the former case only occurs for a narrow range of parameters, whereas the latter case is much more prevalent for a wide range of parameters.

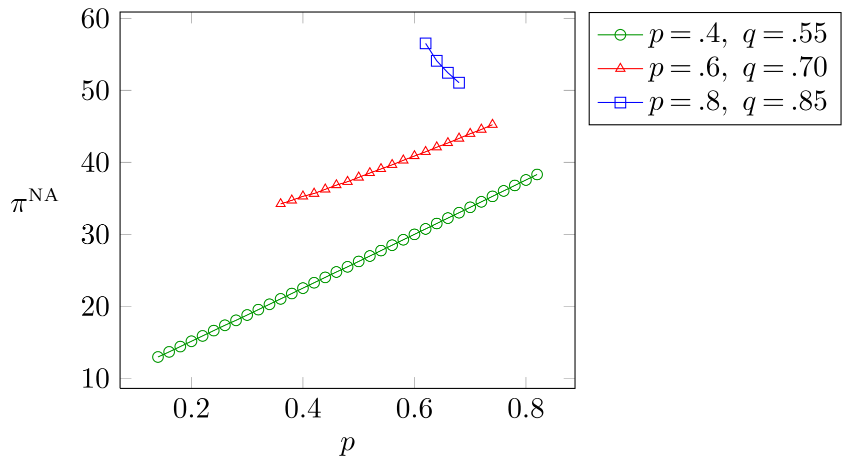


Figure 9 Effects of price p on optimal expected revenue π^{NA} of NA-partition policy (for $T = 100$), plotted for the range of p such that $v_0 \in [v^*, v^{**})$ with the lower price limit corresponding to $v_0 = v^{**}$ and the upper price limit corresponding to $v_0 = v^*$.

7. Conclusion

In this paper, we study the optimal information design for an online platform who aims to maximize its revenue from a time-locked sales campaign. Customers sequentially visit the campaign and they are heterogeneous in that each customer has access to a private signal about the product value. Both the platform and customers are a priori uncertain about the value of the product, but the platform gains informational advantage over customers as the campaign progresses: the platform observes each customer's purchase decision, which is unobservable to other customers. To influence an upcoming customer's inference about the product's value and subsequently her purchase decision, the platform designs its information policy that maps its proprietary up-to-date sales history to a message provided to an upcoming customer.

As a methodological contribution to the emerging dynamic information design literature, we obtain an LP formulation of the platform's problem by reducing the space of information policies to the class of *recommendation policies* that only necessitate three messages (i.e., positive, neutral, and negative recommendations) and by simplifying the representation of the platform's proprietary history through the notion of *net purchase position*. This formulation enables us to efficiently solve

[21] Such a dichotomy in the information provision policy induced by the optimized price is reminiscent of the similar observation in the earlier stream of research on information provision initiated by Lewis and Sappington (1994).

and compute the platform's optimal information policy, and use it to evaluate other heuristic policies.

As a contribution to the revenue management practice, we identify a practically implementable and near-optimal information policy by optimizing over a subclass of recommendation policies, which we refer to as the NA-partition policies. We fully characterize the optimal NA-partition policy, which features minimal randomization of messages, follows a simple structure, and hence is easy to prescribe and interpret. Specifically, the optimal NA-partition policy front-loads the neutral recommendations up to a pre-specified cut-off customer and then back-loads the affirmative recommendations by only promoting the products that are sold sufficiently well before the cut-off customer. We demonstrate numerically that the optimal NA-partition policy can significantly outperform naïve policies commonly used in practice (e.g., the no-disclosure and full-disclosure policies) and delivers a near-optimal revenue.

Finally, our findings provide prescriptive guidelines for online platforms to manage their information communication with customers. Platforms will be significantly better off by switching to the optimal NA-partition policy from the no-disclosure policy for highly rated or heavily discounted products, and from the full-disclosure policy for products with mediocre existing ratings or higher prices. In particular, the optimal NA-partition policy should place the cut-off customer relatively later in the campaign for the latter product category than for the former product category. In general, the optimal NA-partition policy requires only a small fraction of customers (who arrive during the initial segment of the selling horizon) to receive the neutral recommendation.

The general insights and methodology we uncover in this paper are applicable to a wide range of settings (besides time locked sales campaings), whereby social learning is prevalent and can be moderated by an information designer. Examples include social media management (e.g., Facebook, Twitter), crowd-funding web sites (e.g., Indiegogo, Kickstarter), online streaming services (e.g., YouTube, Netflix). Admittedly, our current model has to be modified to account for institutional features specific to these settings. For instance, users on social media may not have equal influence power as they differ in how well they are connected with others; on crowd-funding web sites, project owners can often set and change the reward and its price; customers subscribing an online streaming service may not be short-lived but rather interact with the platforms in a longer term. Examining how such features will affect the platform's information policy (and vice versa) constitutes interesting and potentially fruitful directions for future research.

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Appendix A: Notation

Notation	Definition
T	campaign length
V	product value
p	product price
v_0	prior expectation about product value V
S_t	customer t's private signal
q	precision of customers' private signals
a_t	customer t's purchase decision
m_t	message provided by the platform, to customer t
$\mathcal{M}$	set of all possible messages
H_t	platform's proprietary history up to customer t
$\mathcal{H}$	set of all possible proprietary histories
$(\sigma, \mathcal{M})$	information provision policy
$N(H_t)$	net purchase position, for a given proprietary history H_t
v_n	platform's expectation of the product value V, for a given net purchase position n
r	recommendation policy
$r_t^i(n)$	probability that the platform offers recommendation i to customer t, for a given net purchase position n
$z_t(\cdot)$	belief of a customer, about the net purchase position
$y_t^{i\star}(\cdot)$	probability that the platform offers recommendation i to customer t
ℓ_t^r	total number of neutral customers platform has partitioned before customer t, for a given recommendation policy r
$\zeta(t,\cdot)$	belief of a customer, arriving after t neutral customers, about the net purchase position under a NA-partition policy
τ°	minimum number of neutral customers necessary for the platform to be able to partition affirmative customers
$F(t)$	probability of sending a positive recommendation, after t neutral customers
n^{**}	optimal positive recommendation threshold
n^*	optimal negative recommendation threshold
$\pi^\star$	optimal expected revenue for the platform
π^{NA}	expected revenue of the optimal NA-partition policy
π^{ND}	expected revenue of the no-disclosure policy
π^{FD}	expected revenue of the full-disclosure policy

Appendix B: Proofs in Section 4

Proof of Proposition 1. Since private signals S_t are mutually independent conditional on V with $\mathbb{P}[S_t=1 \mid V=1]=\mathbb{P}[S_t=-1 \mid V=0]=q$, the Bayes rule thus immediately implies that

$$\begin{aligned}
\mathbb{E}[V \mid S_t=1, m_t, \sigma] &= \mathbb{P}[V=1 \mid S_t=1, m_t, \sigma] \\
&= \frac{\mathbb{P}[S_t=1 \mid V=1]\,\mathbb{P}[V=1 \mid m_t,\sigma]}{\mathbb{P}[S_t=1 \mid V=1]\,\mathbb{P}[V=1 \mid m_t,\sigma]+\mathbb{P}[S_t=1 \mid V=0]\,\mathbb{P}[V=0 \mid m_t,\sigma]} \\
&= \frac{q\mathbb{P}[V=1 \mid m_t,\sigma]}{q\mathbb{P}[V=1 \mid m_t,\sigma]+(1-q)\mathbb{P}[V=0 \mid m_t,\sigma]} > \mathbb{P}[V=1 \mid m_t,\sigma]=\mathbb{E}[V \mid m_t,\sigma]
\end{aligned}$$

and similarly

$$\mathbb{E}[V \mid S_t=-1, m_t, \sigma] = \mathbb{P}[V=1 \mid S_t=-1, m_t, \sigma]$$
$$= \frac{(1-q)\mathbb{P}[V=1 \mid m_t, \sigma]}{(1-q)\mathbb{P}[V=1 \mid m_t, \sigma] + q\mathbb{P}[V=0 \mid m_t, \sigma]} < \mathbb{P}[V=1 \mid m_t, \sigma] = \mathbb{E}[V \mid m_t, \sigma],$$

where the inequalities follow by $1/2 < q \le 1$.

Thus, by (2), $a_t = 1$ upon $S_t = -1$ if and only if $\frac{(1-q)\mathbb{P}[V=1 \mid m_t,\sigma]}{(1-q)\mathbb{P}[V=1 \mid m_t,\sigma]+q\mathbb{P}[V=0 \mid m_t,\sigma]} \ge p$, which reduces to $\mathbb{E}[V \mid m_t, \sigma] = \mathbb{P}[V=1 \mid m_t, \sigma] \ge \frac{pq}{pq+(1-p)(1-q)} = v^{**}$. Similarly, $a_t = 1$ upon $S_t = 1$ if and only if $\mathbb{E}[V \mid m_t, \sigma] = \mathbb{P}[V=1 \mid m_t, \sigma] < \frac{p(1-q)}{p(1-q)+(1-p)q} = v^*$. Since it is straightforward to verify (5), the above two conditions thus lead to (4). ■

Lemma B.1. *For an arbitrary information provision policy* $\left(\tilde{\sigma}, \widetilde{\mathcal{M}}\right)$, *define a mapping* $\varphi : \widetilde{\mathcal{M}} \to \mathcal{M} := \{1, 0, -1\}$ *as follows:*

$$\varphi(\widetilde{m}_t) := \begin{cases} 1, & \text{if } \mathbb{E}[V \mid \widetilde{m}_t, \tilde{\sigma}] \ge v^{**}, \\ 0, & \text{if } v^* \le \mathbb{E}[V \mid \widetilde{m}_t, \tilde{\sigma}] < v^{**}, \\ -1, & \text{if } \mathbb{E}[V \mid \widetilde{m}_t, \tilde{\sigma}] \le v^*. \end{cases} \tag{B.1}$$

For a platform's proprietary history $\widetilde{H}_t = \{(\widetilde{m}_s, a_s) : s < t\}$ *under* $\left(\tilde{\sigma}, \widetilde{\mathcal{M}}\right)$, *denote* $\varphi(\widetilde{H}_t) := \{(\varphi(\widetilde{m}_s), a_s) : s < t\}$ *with slight abuse of notation. Then,*

$$\mathbb{E}\left[V \,\middle|\, \widetilde{H}'_t, \tilde{\sigma}\right] = \mathbb{E}\left[V \,\middle|\, \widetilde{H}''_t, \tilde{\sigma}\right], \quad \text{if } \varphi(\widetilde{H}'_t) = \varphi(\widetilde{H}''_t). \tag{B.2}$$

Furthermore, platform's expectation of the product value evolves as follows

$$\mathbb{E}\left[V \,\middle|\, \widetilde{H}_{t+1}, \tilde{\sigma}\right] = \begin{cases} \mathbb{E}\left[V \,\middle|\, \widetilde{H}_t, \tilde{\sigma}\right], & \text{if } \varphi(\widetilde{m}_t) = \pm 1 \text{ and } \widetilde{H}_{t+1} = \widetilde{H}_t \cup (\widetilde{m}_t, \varphi(\widetilde{m}_t)), \\ \frac{q\mathbb{P}\left[V=1 \,\middle|\, \widetilde{H}_t, \tilde{\sigma}\right]}{q\mathbb{P}\left[V=1 \,\middle|\, \widetilde{H}_t, \tilde{\sigma}\right] + (1-q)\mathbb{P}\left[V=0 \,\middle|\, \widetilde{H}_t, \tilde{\sigma}\right]}, & \text{if } \varphi(\widetilde{m}_t) = 0 \text{ and } \widetilde{H}_{t+1} = \widetilde{H}_t \cup (\widetilde{m}_t, 1), \\ \frac{(1-q)\mathbb{P}\left[V=1 \,\middle|\, \widetilde{H}_t, \tilde{\sigma}\right]}{(1-q)\mathbb{P}\left[V=1 \,\middle|\, \widetilde{H}_t, \tilde{\sigma}\right] + q\mathbb{P}\left[V=0 \,\middle|\, \widetilde{H}_t, \tilde{\sigma}\right]}, & \text{if } \varphi(\widetilde{m}_t) = 0 \text{ and } \widetilde{H}_{t+1} = \widetilde{H}_t \cup (\widetilde{m}_t, -1). \end{cases} \tag{B.3}$$

with $\mathbb{E}[V \mid \tilde{\sigma}] = \mathbb{P}[V=1 \mid \tilde{\sigma}] = v_0$.

Proof of Lemma B.1. By Proposition 1 and definition of φ in (B.1), customer t, who receives $\widetilde{m}_t$ with $\varphi(\widetilde{m}_t) = \pm 1$, makes purchase decision $\widetilde{a}_t = \varphi(\widetilde{m}_t) = \pm 1$, regardless of S_t. Thus, platform does not infer any new information in addition to the inference made from the previous customers purchase decisions and hence its expectation of the product value does not alter, establishing the first line in (B.3). Again by Proposition 1 and (B.1), customer t, who receives $\widetilde{m}_t$ where $\varphi(\widetilde{m}_t) = 0$, makes purchase decision $\widetilde{a}_t = S_t$. Thus, the second and third lines in (B.3) follow from the Bayes rule:

$$\mathbb{E}\left[V \,\middle|\, \widetilde{H}_t \cup (\widetilde{m}_t, \widetilde{a}_t), \varphi(\widetilde{m}_t) = 0, \tilde{\sigma}\right] = \mathbb{P}\left[V=1 \,\middle|\, \widetilde{H}_t \cup (\widetilde{m}_t, \widetilde{a}_t), \varphi(\widetilde{m}_t) = 0, \tilde{\sigma}\right]$$
$$= \mathbb{P}\left[V=1 \,\middle|\, \widetilde{H}_t, S_t = \widetilde{a}_t, \tilde{\sigma}\right]$$
$$= \frac{\mathbb{P}[S_t = \widetilde{a}_t \mid V=1]\,\mathbb{P}\left[V=1 \,\middle|\, \widetilde{H}_t, \tilde{\sigma}\right]}{\mathbb{P}[S_t = \widetilde{a}_t \mid V=1]\,\mathbb{P}\left[V=1 \,\middle|\, \widetilde{H}_t, \tilde{\sigma}\right] + \mathbb{P}[S_t = \widetilde{a}_t \mid V=0]\,\mathbb{P}\left[V=0 \,\middle|\, \widetilde{H}_t, \tilde{\sigma}\right]},$$

where $\mathbb{P}[S_t = 1 \mid V=1] = \mathbb{P}[S_t = -1 \mid V=0] = q$ by assumption.

We now demonstrate (B.2) by induction. For $t = 1$, since $\widetilde{H}_1 = \emptyset$ and $\mathbb{E}[V \mid \tilde{\sigma}] = v_0$, (B.2) holds. Now suppose (B.2) holds for t. Let $\widetilde{H}'_{t+1}, \widetilde{H}''_{t+1}$ be two platform's proprietary histories such that $\varphi(\widetilde{H}'_{t+1}) = \varphi(\widetilde{H}''_{t+1})$, which implies that $\varphi(\widetilde{H}'_t) = \varphi(\widetilde{H}''_t)$ and $\varphi(\widetilde{m}'_t) = \varphi(\widetilde{m}''_t)$. Thus, the induction hypothesis immediately implies that $\mathbb{E}\left[V \,\middle|\, \widetilde{H}'_t, \tilde{\sigma}\right] = \mathbb{E}\left[V \,\middle|\, \widetilde{H}''_t, \tilde{\sigma}\right]$ and, further by (B.3), we have (B.2) holds for $t+1$. ■

Proof of Corollary 1. We prove by induction the following property for $v_0 \notin [v^*, v^{**})$: under any arbitrary information provision policy σ,

$$\mathbb{E}[V \mid m_t, \sigma] = \mathbb{E}[V \mid H_t, \sigma] = v_0, \tag{B.4}$$

for any m_t and H_t such that $\sigma(m_t \mid H_t) > 0$ and $\mathbb{P}[H_t \mid \sigma] > 0$, for all $t \in \{1, \cdots, T\}$.

For $t = 1$, by Bayes rule we have for any m_1

$$\begin{aligned}\mathbb{E}[V \mid m_1, \sigma] = \mathbb{P}[V = 1 \mid m_1, \sigma] =& \frac{\sum_{H_1} \sigma(m_1 \mid H_1) \mathbb{P}[V = 1 \mid H_1, \sigma] \mathbb{P}[H_1 \mid \sigma]}{\sum_{H_1} \sigma(m_1 \mid H_1) \mathbb{P}[H_1 \mid \sigma]} \\ =& \mathbb{P}[V = 1 \mid H_1, \sigma] = \mathbb{E}[V] = v_0,\end{aligned}$$

where the third and fourth equalities follow from the fact that $H_1 = \emptyset$. Hence, (B.4) holds for $t = 1$. Now suppose (B.4) holds for an arbitrary t. Again by Bayes rule, we have for any m_{t+1}

$$\mathbb{E}[V \mid m_{t+1}, \sigma] = \mathbb{P}[V = 1 \mid m_{t+1}, \sigma] = \frac{\sum_{H_{t+1}} \sigma(m_{t+1} \mid H_{t+1}) \mathbb{P}[V = 1 \mid H_{t+1}, \sigma] \mathbb{P}[H_{t+1} \mid \sigma]}{\sum_{H_{t+1}} \sigma(m_{t+1} \mid H_{t+1}) \mathbb{P}[H_{t+1} \mid \sigma]} \tag{B.5}$$

Notice for each H_{t+1} with $\mathbb{P}[H_{t+1} \mid \sigma] > 0$, by our induction hypothesis and Proposition 1, that we either have $H_{t+1} = H_t \cup (m_t, 1)$ or $H_{t+1} = H_t \cup (m_t, -1)$ conditional on whether $v_0 \geq v^{**}$ or $v_0 < v^*$, respectively. Consequently, again by our induction hypothesis and (B.3) of Lemma B.1

$$\mathbb{P}[V = 1 \mid H_{t+1}, \sigma] = \mathbb{E}[V \mid H_{t+1}, \sigma] = \mathbb{E}[V \mid H_t, \sigma] = v_0,$$

for all m_t and H_t that satisfy $\sigma(m_t \mid H_t) > 0$ and $\mathbb{P}[H_t \mid \sigma] > 0$. Hence, (B.5) becomes $\mathbb{E}[V \mid m_{t+1}, \sigma] = v_0$ for all m_{t+1} with $\mathbb{P}[m_{t+1} \mid \sigma] > 0$. Therefore, (B.4) also holds for $t + 1$.

Utilizing (B.4) and Proposition 1 it is straightforward to see for all $t \in \{1, \cdots, T\}$, $\mathbb{P}[a_t = 1 \mid \sigma] = 1$ when $v_0 \geq v^{**}$ and $\mathbb{P}[a_t = 1 \mid \sigma] = 0$ when $v_0 < v^*$. This completes the proof. ■

Proof of Proposition 2. For an arbitrary information provision policy $\left(\tilde{\sigma}, \widetilde{\mathcal{M}}\right)$, we now define a new information provision policy $(\sigma, \mathcal{M})$ as

$$\sigma(m_t \mid H_t) := \frac{\sum\limits_{\varphi(\widetilde{H}_t) = H_t, \varphi(\widetilde{m}_t) = m_t} \tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t\right) \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right]}{\sum\limits_{\varphi(\widetilde{H}_t) = H_t} \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right]}, \tag{B.6}$$

for any $m_t \in \mathcal{M}$ and $H_t = \{(m_s, a_s) : m_s \in \mathcal{M}, a_s \in \{-1, 1\}, s < t\}$.

We then show that the information provision policy $(\sigma, \mathcal{M})$ defined above satisfies ($\mathrm{IC}^{\sigma}_{-1}$), ($\mathrm{IC}^{\sigma}_{0}$), and ($\mathrm{IC}^{\sigma}_{1}$), and hence is a recommendation policy. By rule of total probability, we first have

$$\begin{aligned}\mathbb{P}[m_t \mid \sigma] =& \sum_{H_t} \sigma(m_t \mid H_t) \mathbb{P}[H_t \mid \sigma] \\ \text{by (B.6)} \quad =& \sum_{H_t} \frac{\sum\limits_{\varphi(\widetilde{H}_t) = H_t, \varphi(\widetilde{m}_t) = m_t} \tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t\right) \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right]}{\sum\limits_{\varphi(\widetilde{H}_t) = H_t} \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right]} \sum_{\varphi(\widetilde{H}_t) = H_t} \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right] \\ =& \sum_{\widetilde{H}_t, \varphi(\widetilde{m}_t) = m_t} \tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t\right) \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right] = \sum_{\varphi(\widetilde{m}_t) = m_t} \mathbb{P}[\widetilde{m}_t \mid \tilde{\sigma}],\end{aligned} \tag{B.7}$$

and similarly,

$$
\begin{aligned}
\mathbb{P}[m_t, V=1 \mid \sigma] &= \sum_{H_t} \sigma(m_t \mid H_t)\, \mathbb{P}[V=1, H_t \mid \sigma] \\
&= \sum_{H_t} \sigma(m_t \mid H_t) \left\{ \sum_{\varphi(\widetilde{H}_t)=H_t} \mathbb{P}\left[V=1 \,\middle|\, \widetilde{H}_t, \tilde{\sigma}\right] \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right] \right\} \\
\text{by (B.6)} \quad &= \sum_{H_t} \frac{\sum_{\varphi(\widetilde{H}_t)=H_t, \varphi(\widetilde{m}_t)=m_t} \tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t\right) \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right]}{\sum_{\varphi(\widetilde{H}_t)=H_t} \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right]} \left\{ \sum_{\varphi(\widetilde{H}_t)=H_t} \mathbb{P}\left[V=1 \,\middle|\, \widetilde{H}_t, \tilde{\sigma}\right] \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right] \right\} \\
&= \sum_{H_t} \sum_{\varphi(\widetilde{H}_t)=H_t, \varphi(\widetilde{m}_t)=m_t} \mathbb{P}\left[V=1 \,\middle|\, \widetilde{H}_t, \tilde{\sigma}\right] \tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t\right) \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right] \\
&= \sum_{\varphi(\widetilde{m}_t)=m_t} \sum_{\widetilde{H}_t} \mathbb{P}\left[V=1 \,\middle|\, \widetilde{H}_t, \tilde{\sigma}\right] \tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t\right) \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right],
\end{aligned}
$$

where the fourth equality, utilizing the fact that $V \in \{0,1\}$ is a binary random variable, follows by (B.2). Then, the Bayes rule yields

$$
\mathbb{E}[V \mid m_t, \sigma] = \mathbb{P}[V=1 \mid m_t, \sigma] = \frac{\sum_{\varphi(\widetilde{m}_t)=m_t} \sum_{\widetilde{H}_t} \mathbb{P}\left[V=1 \,\middle|\, \widetilde{H}_t, \tilde{\sigma}\right] \tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t\right) \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right]}{\sum_{\varphi(\widetilde{m}_t)=m_t} \mathbb{P}[\widetilde{m}_t \mid \tilde{\sigma}]}. \tag{B.8}
$$

On the other hand, the Bayes rule also yields

$$
\mathbb{E}[V \mid \widetilde{m}_t, \tilde{\sigma}] = \mathbb{P}[V=1 \mid \widetilde{m}_t, \tilde{\sigma}] = \frac{\sum_{\widetilde{H}_t} \mathbb{P}\left[V=1 \,\middle|\, \widetilde{H}_t, \tilde{\sigma}\right] \tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t\right) \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right]}{\mathbb{P}[\widetilde{m}_t \mid \tilde{\sigma}]}. \tag{B.9}
$$

For any $\widetilde{m}_t$ such that $\varphi(\widetilde{m}_t)=0$, (B.1) implies that $v^* \le \mathbb{E}[V \mid \widetilde{m}_t, \tilde{\sigma}] < v^{**}$, which, by (B.9), is equivalent to

$$
v^* \mathbb{P}[\widetilde{m}_t \mid \tilde{\sigma}] \le \sum_{\widetilde{H}_t} \mathbb{P}\left[V=1 \,\middle|\, \widetilde{H}_t, \tilde{\sigma}\right] \tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t\right) \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right] < v^{**} \mathbb{P}[\widetilde{m}_t \mid \tilde{\sigma}].
$$

Further by (B.8), we have $\mathbb{E}[V \mid m_t = 0, \sigma] = \mathbb{P}[V=1 \mid m_t=0, \sigma] \in [v^*, v^{**})$, establishing ($\text{IC}_0^\sigma$). By the same token, we can also establish (IC_{-1}^σ) and (IC_1^σ).

Finally, we demonstrate that $(\sigma, \mathcal{M})$ and $\left(\tilde{\sigma}, \widetilde{\mathcal{M}}\right)$ induces the same (ex ante) probability of purchase for each customer and generates the same expected revenue for the platform. By Proposition 1 and (B.1), customer t's purchase decision under $\left(\tilde{\sigma}, \widetilde{\mathcal{M}}\right)$ is

$$
\tilde{a}_t = \varphi(\widetilde{m}_t) + (1 - |\varphi(\widetilde{m}_t)|)\, S_t. \tag{B.10}
$$

And customer t's purchase decision under $(\sigma, \mathcal{M})$ is

$$
a_t = m_t + (1 - |m_t|)\, S_t. \tag{B.11}
$$

Hence, by (3), the platform's expected revenue under $\left(\tilde{\sigma}, \widetilde{\mathcal{M}}\right)$ is given by $\frac{pT}{2} + \frac{p}{2}\sum_{t=1}^T \mathbb{E}[\tilde{a}_t \mid \tilde{\sigma}]$, and the platform's expected revenue under $(\sigma, \mathcal{M})$ is given by $\frac{pT}{2} + \frac{p}{2}\sum_{t=1}^T \mathbb{E}[a_t \mid \sigma]$. We now demonstrate these revenues are equal by showing

$$
\mathbb{E}[\tilde{a}_t \mid \tilde{\sigma}] = \mathbb{E}[a_t \mid \sigma] \text{ for all } t. \tag{B.12}
$$

On one hand, by (B.10)

$$\begin{aligned}\mathbb{E}\left[\tilde{a}_t \mid \tilde{\sigma}\right] &= \mathbb{E}\left[\varphi(\widetilde{m}_t) + (1 - |\varphi(\widetilde{m}_t)|)\, S_t \mid \tilde{\sigma}\right] \\ &= 1 \sum_{\varphi(\widetilde{m}_t)=1} \mathbb{P}\left[\widetilde{m}_t \mid \tilde{\sigma}\right] + (-1) \sum_{\varphi(\widetilde{m}_t)=-1} \mathbb{P}\left[\widetilde{m}_t \mid \tilde{\sigma}\right] + \sum_{\varphi(\widetilde{m}_t)=0} \mathbb{E}\left[S_t \mid \widetilde{m}_t, \tilde{\sigma}\right] \mathbb{P}\left[\widetilde{m}_t \mid \tilde{\sigma}\right]. \end{aligned} \tag{B.13}$$

On the other hand, by (B.11)

$$\begin{aligned}\mathbb{E}\left[a_t \mid \sigma\right] &= \mathbb{E}\left[m_t + (1 - |m_t|)\, S_t \mid \sigma\right] \\ &= 1\mathbb{P}\left[m_t = 1 \mid \sigma\right] + (-1)\mathbb{P}\left[m_t = -1 \mid \sigma\right] + \mathbb{E}\left[S_t \mid m_t = 0, \sigma\right] \mathbb{P}\left[m_t = 0 \mid \sigma\right]. \end{aligned} \tag{B.14}$$

By (B.7), the first two terms in (B.13) are equal to the first two terms in (B.14) respectively, i.e., $\sum_{\varphi(\widetilde{m}_t)=1} \mathbb{P}\left[\widetilde{m}_t \mid \tilde{\sigma}\right] = \mathbb{P}\left[m_t = 1 \mid \sigma\right]$ and $\sum_{\varphi(\widetilde{m}_t)=-1} \mathbb{P}\left[\widetilde{m}_t \mid \tilde{\sigma}\right] = \mathbb{P}\left[m_t = -1 \mid \sigma\right]$. Below, we demonstrate the last term in (B.13) is also equal to that in (B.14), thus completing the proof. To that end, denote $\widetilde{u}_{\widetilde{H}_t} := q\mathbb{P}\left[V = 1 \,\middle|\, \widetilde{H}_t, \tilde{\sigma}\right] + (1-q)\mathbb{P}\left[V = 0 \,\middle|\, \widetilde{H}_t, \tilde{\sigma}\right]$ and $u_{H_t} := q\mathbb{P}\left[V = 1 \mid H_t, \sigma\right] + (1-q)\mathbb{P}\left[V = 0 \mid H_t, \sigma\right]$. Then, the last term in (B.13) can be written as

$$\begin{aligned}\sum_{\varphi(\widetilde{m}_t)=0} \mathbb{E}\left[S_t \mid \widetilde{m}_t, \tilde{\sigma}\right] \mathbb{P}\left[\widetilde{m}_t \mid \tilde{\sigma}\right] &= \sum_{\varphi(\widetilde{m}_t)=0} \sum_{\widetilde{H}_t} \left[1\widetilde{u}_{\widetilde{H}_t} + (-1)(1 - \widetilde{u}_{\widetilde{H}_t})\right] \tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t\right) \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right] \\ &= \sum_{\varphi(\widetilde{m}_t)=0} \sum_{\widetilde{H}_t} \left[2\widetilde{u}_{\widetilde{H}_t} - 1\right] \tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t\right) \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right]. \end{aligned}$$

Similarly, the last term in (B.14) can be rewritten as

$$\begin{aligned}\mathbb{E}\left[S_t \mid m_t = 0, \sigma\right] \mathbb{P}\left[m_t = 0 \mid \sigma\right] &= \sum_{H_t} \left[1u_{H_t} + (-1)(1 - u_{H_t})\right] \sigma\left(m_t = 0 \mid H_t\right) \mathbb{P}\left[H_t \mid \sigma\right] \\ &= \sum_{H_t} \left[2u_{H_t} - 1\right] \sigma\left(m_t = 0 \mid H_t\right) \mathbb{P}\left[H_t \mid \sigma\right] \\ &= \sum_{H_t} \left[2u_{H_t} - 1\right] \frac{\sum_{\varphi(\widetilde{H}_t)=H_t, \varphi(\widetilde{m}_t)=0} \tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t\right) \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right]}{\sum_{\varphi(\widetilde{H}_t)=H_t} \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right]} \mathbb{P}\left[H_t \mid \sigma\right] \\ &= \sum_{H_t} \left[2u_{H_t} - 1\right] \frac{\sum_{\varphi(\widetilde{H}_t)=H_t, \varphi(\widetilde{m}_t)=0} \tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t\right) \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right]}{\sum_{\varphi(\widetilde{H}_t)=H_t} \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right]} \sum_{\varphi(\widetilde{H}_t)=H_t} \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right] \\ &= \sum_{H_t} \sum_{\varphi(\widetilde{H}_t)=H_t, \varphi(\widetilde{m}_t)=0} \left[2u_{H_t} - 1\right] \tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t\right) \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right] \\ &= \sum_{\varphi(\widetilde{m}_t)=0} \sum_{\widetilde{H}_t} \left[2\widetilde{u}_{\widetilde{H}_t} - 1\right] \tilde{\sigma}\left(\widetilde{m}_t \,\middle|\, \widetilde{H}_t\right) \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right] \\ &= \sum_{\varphi(\widetilde{m}_t)=0} \mathbb{E}\left[S_t \mid \widetilde{m}_t, \tilde{\sigma}\right] \mathbb{P}\left[\widetilde{m}_t \mid \tilde{\sigma}\right], \end{aligned}$$

where the third equality follows from (B.6) and the conversion of u_{H_t} to $\widetilde{u}_{\widetilde{H}_t}$ in the sixth equality follows from (B.2) and the fact that $V \in \{0, 1\}$ is a binary random variable. ■

Proof of Proposition 3. Under a recommendation policy σ, (B.3) implies that the platform's expectation of the product value evolves according to

$$\mathbb{E}\left[V \mid H_{t+1},\sigma\right]=\begin{cases}\mathbb{E}\left[V \mid H_t,\sigma\right], & \text{if } H_{t+1}=H_t\cup\pm(1,1)\,,\\ \frac{q\mathbb{P}[V=1\mid H_t,\sigma]}{q\mathbb{P}[V=1\mid H_t,\sigma]+(1-q)\mathbb{P}[V=0\mid H_t,\sigma]}, & \text{if } H_{t+1}=H_t\cup(0,1)\,,\\ \frac{(1-q)\mathbb{P}[V=1\mid H_t,\sigma]}{(1-q)\mathbb{P}[V=1\mid H_t,\sigma]+q\mathbb{P}[V=0\mid H_t,\sigma]}, & \text{if } H_{t+1}=H_t\cup(0,-1)\,,\end{cases} \tag{B.15}$$

with $\mathbb{E}\left[V\mid\sigma\right]=\mathbb{P}\left[V=1\mid\sigma\right]=v_0$. We now prove (6) by induction. For $t=1$, since $H_1=\emptyset$ and hence $N(H_1)=0$, thus (6) follows as $\mathbb{E}\left[V\mid\sigma\right]=v_0$. Suppose (6) holds for any arbitrary t (i.e., induction hypothesis). Then, under a recommendation policy σ, we have the following three cases for $t+1$:

- If $H_{t+1}=H_t\cup\pm(1,1)$, (B.15) implies that $\mathbb{E}\left[V\mid H_{t+1},\sigma\right]=\mathbb{E}\left[V\mid H_t,\sigma\right]$ and hence (6) holds for $t+1$ by induction hypothesis and by noting that $N(H_{t+1})=N(H_t)$ according to (7).
- If $H_{t+1}=H_t\cup(0,1)$, (B.15) and induction hypothesis implies that

$$\mathbb{E}\left[V\mid H_{t+1},\sigma\right]=\mathbb{P}\left[V=1\mid H_{t+1},\sigma\right]=\frac{qv_0}{qv_0+(1-q)(1-v_0)\left(\frac{1-q}{q}\right)^{N(H_t)}}=\frac{v_0}{v_0+(1-v_0)\left(\frac{1-q}{q}\right)^{N(H_t)+1}},$$

where the second equality follows by noting that $N(H_{t+1})=N(H_t)+1$ according to (7), hence (6) holds for $t+1$.
- If $H_{t+1}=H_t\cup(0,-1)$, (B.15) and induction hypothesis implies that

$$\mathbb{E}\left[V\mid H_{t+1},\sigma\right]=\mathbb{P}\left[V=1\mid H_{t+1},\sigma\right]=\frac{(1-q)v_0}{(1-q)v_0+q(1-v_0)\left(\frac{1-q}{q}\right)^{N(H_t)}}=\frac{v_0}{v_0+(1-v_0)\left(\frac{1-q}{q}\right)^{N(H_t)-1}},$$

where the second equality follows by noting that $N(H_{t+1})=N(H_t)-1$ according to (7), hence (6) holds for $t+1$. ■

Proof of Proposition 4. By the definition of recommendation policy σ, the platform's proprietary history evolves according to

$$\mathbb{P}\left[H_{t+1}\mid H_t,m_t,\sigma\right]=\begin{cases}1, & \text{if } m_t=\pm1 \text{ and } H_{t+1}=H_t\cup(m_t,m_t)\,,\\ q\mathbb{P}\left[V=1\mid H_t,\sigma\right]+(1-q)\mathbb{P}\left[V=0\mid H_t,\sigma\right], & \text{if } m_t=0 \text{ and } H_{t+1}=H_t\cup(0,1)\,,\\ (1-q)\mathbb{P}\left[V=1\mid H_t,\sigma\right]+q\mathbb{P}\left[V=0\mid H_t,\sigma\right], & \text{if } m_t=0 \text{ and } H_{t+1}=H_t\cup(0,-1)\,,\end{cases}$$

which, by the characterization of $\mathbb{E}\left[V\mid H_t,\sigma\right]$ (equivalently, $\mathbb{P}\left[V=1\mid H_t,\sigma\right]$ as $V\in\{0,1\}$ is a binary random variable) in (6) and the definition of v_n in (8), translates to

$$\mathbb{P}\left[N(H_{t+1})\mid N(H_t)=k,m_t\right]=\begin{cases}1, & \text{if } m_t=\pm1 \text{ and } N(H_{t+1})=k,\\ qv_k+(1-q)(1-v_k)=u_k, & \text{if } m_t=0 \text{ and } N(H_{t+1})=k+1,\\ (1-q)v_k+q(1-v_k)=1-u_k, & \text{if } m_t=0 \text{ and } N(H_{t+1})=k-1,\end{cases} \tag{B.16}$$

for any k. Total probability rule implies that

$$\begin{aligned}\mathbb{P}\left[N(H_{t+1})=n\mid r\right]&=\sum_{k=n-1}^{n+1}\mathbb{P}\left[N(H_{t+1})=n\mid N(H_t)=k,r\right]\mathbb{P}\left[N(H_t)=k\mid r\right]\\ &=\sum_{k=n-1}^{n+1}\sum_{i=-1}^{1}\mathbb{P}\left[N(H_{t+1})=n\mid N(H_t)=k,m_t=i\right]\mathbb{P}\left[m_t=i\mid N(H_t)=k,r\right]z_t(k)\\ &=\sum_{k=n-1}^{n+1}\sum_{i=-1}^{1}\mathbb{P}\left[N(H_{t+1})=n\mid N(H_t)=k,m_t=i\right]r_t^i(k)z_t(k),\end{aligned}$$

which, by substituting $\mathbb{P}\left[N(H_{t+1})=n\mid N(H_t)=k,m_t=i\right]$ with (B.16), yields (N). ■

Proof of Proposition 5. On one hand, let $z_t^\star(n)$ and $r_t^{i\star}(n)$ for $i\in\{1,0,-1\}$, $n=-T+1,\cdots,T-1$ and $t=1,\cdots,T$ be the optimal solution to (9). We verify below that $y_t^{i\star}(n):=z_t^\star(n)r_t^{i\star}(n)$ for $i\in\{1,0,-1\}$, $n=-T+1,\cdots,T-1$ and $t=1,\cdots,T$ is a feasible solution to (11). First, notice by (R) and (N) that, $y_t^{i\star}(n)=z_t^\star(n)r_t^{i\star}(n)\geq 0$ for $i\in\{1,0,-1\}$, $n=-T+1,\cdots,T-1$ and $t=1,\cdots,T$. Second, utilizing formulation $y_t^{i\star}(n)=z_t^\star(n)r_t^{i\star}(n)$, we have

$$\begin{aligned}
\sum_{i\in\{1,0,-1\}} y_t^{i\star}(n) =& z_t^\star(n)\underbrace{\left[r_t^{1\star}(n)+r_t^{0\star}(n)+r_t^{-1\star}(n)\right]}_{=1 \text{ by (R)}}\\
=&\left[z_t^\star(n)-z_{t-1}^\star(n)\right]+\left[z_{t-1}^\star(n)-z_{t-2}^\star(n)\right]+\cdots+\left[z_2^\star(n)-z_1^\star(n)\right]+z_1^\star(n)\\
\text{by (N)}\quad =&u_{n-1}\sum_{s=1}^{t-1}z_s^\star(n-1)r_s^{0\star}(n-1)+(1-u_{n+1})\sum_{s=1}^{t-1}z_s^\star(n+1)r_s^{0\star}(n+1)-\sum_{s=1}^{t-1}z_s^\star(n)r_s^{0\star}(n)+\mathbb{1}[n=0]\\
=&u_{n-1}\sum_{s=1}^{t-1}y_s^{0\star}(n-1)+(1-u_{n+1})\sum_{s=1}^{t-1}y_s^{0\star}(n+1)-\sum_{s=1}^{t-1}y_s^{0\star}(n)+\mathbb{1}[n=0],
\end{aligned}$$

for all $n=-T+1,\cdots,T-1$ and $t=1,\cdots,T$. Remaining set of constraints are also established from utilizing the formulation $y_t^{i\star}(n)=z_t^\star(n)r_t^{i\star}(n)$ and (IC_1^r)-(IC_{-1}^r), such that

$$\begin{aligned}
\text{by }(\text{IC}_1^r)\quad &=\sum_{n=-T+1}^{T-1}(v_n-v^{**})y_t^{1\star}(n)=\sum_n(v_n-v^{**})z_t^\star(n)r_t^{1\star}(n)\geq 0,\\
\text{by }(\text{IC}_0^r)\quad &=\sum_{n=-T+1}^{T-1}(v_n-v^{**})y_t^{0\star}(n)=\sum_n(v_n-v^{**})z_t^\star(n)r_t^{0\star}(n)\leq 0,\\
\text{by }(\text{IC}_0^r)\quad &=\sum_{n=-T+1}^{T-1}(v_n-v^{*})y_t^{0\star}(n)=\sum_n(v_n-v^{*})z_t^\star(n)r_t^{0\star}(n)\geq 0,\\
\text{by }(\text{IC}_{-1}^r)\quad &=\sum_{n=-T+1}^{T-1}(v_n-v^{*})y_t^{-1\star}(n)=\sum_n(v_n-v^{*})z_t^\star(n)r_t^{-1\star}(n)\leq 0.
\end{aligned}$$

Furthermore, it is straightforward to see that

$$\pi^\star=p\sum_{t=1}^{T}\sum_n z_t^\star(n)\left[r_t^{1\star}(n)+u_nr_t^{0\star}(n)\right]=p\sum_{t=1}^{T}\sum_{n=-T+1}^{T-1}\left[y_t^{1\star}(n)+u_ny_t^{0\star}(n)\right]\leq\text{optimal objective value of (11)}. \tag{B.17}$$

On the other hand, let $y_t^{i\star}(n)$ for $i\in\{1,0,-1\}$, $n=-T+1,\cdots,T-1$ and $t=1,\cdots,T$ be the optimal solution to (11). We can verify below that $z_t^\star(n)$ and $r_t^{i\star}(n)$ for $i\in\{1,0,-1\}$ defined by (10) (with $r_t^\star(n)$ being any vector satisfying (R) for $z_t^\star(n)=0$) is a feasible solution to (9). First, notice that (R) is also satisfied for $z_t^\star(n)>0$, following from (10) and the fact that $y_t^{i\star}(n)\geq 0$ for all $i\in\{1,0,-1\}$, $n=-T+1,\cdots,T-1$ and $t=1,\cdots,T$,

$$r_t^{1\star}(n)+r_t^{0\star}(n)+r_t^{-1\star}(n)=\sum_{i\in\{1,0,-1\}}\frac{y_t^{i\star}(n)}{z_t^\star(n)}=\frac{z_t^\star(n)}{z_t^\star(n)}=1.$$

Second, by utilizing the fact that $z_1(0)=1$ and $z_1(n)=0$ for all $n\neq 0$ and (10), we have

$$z_t^\star(n)-z_1^\star(n)=u_{n-1}\sum_{s=1}^{t-1}y_s^{0\star}(n-1)+(1-u_{n+1})\sum_{s=1}^{t-1}y_s^{0\star}(n+1)-\sum_{s=1}^{t-1}y_s^{0\star}(n),$$

which can equivalently be written as

$$\left[z_t^\star(n)-z_{t-1}^\star(n)\right]+\left[z_{t-1}^\star(n)-z_{t-2}^\star(n)\right]+\cdots+\left[z_2^\star(n)-z_1^\star(n)\right]=$$

$$u_{n-1}\sum_{s=1}^{t-1} z_s^\star(n-1)r_s^{0\star}(n-1)+(1-u_{n+1})\sum_{s=1}^{t-1} z_s^\star(n+1)r_s^{0\star}(n+1)-\sum_{s=1}^{t-1} z_s^\star(n)r_s^{0\star}(n),$$

that immediately establishes the feasibility of $z_t^\star(n)$ and $r_t^{i\star}(n)$ for $i\in\{1,0,-1\}$ defined by (10), for constraint (N). Again by (10), we also have

$$\begin{aligned}
\sum_n (v_n-v^{**})z_t^\star(n)r_t^{1\star}(n) &= \sum_{n=-T+1}^{T-1}(v_n-v^{**})y_t^{1\star}(n)\ge 0,\\
\sum_n (v_n-v^{**})z_t^\star(n)r_t^{0\star}(n) &= \sum_{n=-T+1}^{T-1}(v_n-v^{**})y_t^{0\star}(n)\le 0,\\
\sum_n (v_n-v^{*})z_t^\star(n)r_t^{0\star}(n) &= \sum_{n=-T+1}^{T-1}(v_n-v^{*})y_t^{0\star}(n)\ge 0,\\
\sum_n (v_n-v^{*})z_t^\star(n)r_t^{-1\star}(n) &= \sum_{n=-T+1}^{T-1}(v_n-v^{*})y_t^{-1\star}(n)\le 0.
\end{aligned}$$

establishing the IC constraints (IC_1^r)-(IC_{-1}^r), respectively. Furthermore, it is straightforward to see that

$$\text{optimal objective value of (11)} = p\sum_{t=1}^{T}\sum_{n=-T+1}^{T-1}\left[y_t^{1\star}(n)+u_n y_t^{0\star}(n)\right]=p\sum_{t=1}^{T}\sum_n z_t^\star(n)\left[r_t^{1\star}(n)+u_n r_t^{0\star}(n)\right]\le\pi^\star \tag{B.18}$$

which, combined with (B.17), implies that optimal objective value of (11) $=\pi^\star$, establishing the theorem. ■

Appendix C: Proofs in Sections 5 and 6

Proof of Proposition 6. We prove this proposition by induction. It is straightforward to see (12) holds for customer $t=1$, since $\ell_1^r=0$ by definition and we have $z_1(0)=1, z_1(n)=0\ \forall n\neq 0$ by (N).

Now, suppose that (12) holds for customer t (i.e., induction hypothesis) and $\ell_t^r=s$. If $r_t^0(n)\equiv 1$ or equivalently $\ell_{t+1}^r=\ell_t^r+1=s+1$, for an arbitrary net purchase position n with $n\equiv s+1\ (\text{mod}2)$ and $|n|\le s+1$, we have

$$z_{t+1}(n)=[\underbrace{qv_{n-1}+(1-q)(1-v_{n-1})}_{u_{n-1}}]z_t(n-1)+[\underbrace{(1-q)v_{n+1}+q(1-v_{n+1})}_{1-u_{n+1}}]z_t(n+1), \tag{C.1}$$

by (N) as $r_t^0(n)=1\ \forall n$. For $|n|\le s-1$, substituting the definition of v_n in (8) into (C.1) and using our induction hypothesis, we obtain

$$\begin{aligned}
z_{t+1}(n)=&\frac{v_0q^n+(1-v_0)(1-q)^n}{v_0q^{n-1}+(1-v_0)(1-q)^{n-1}}\binom{s}{\frac{s+n-1}{2}}\left[v_0q^{\frac{s+n-1}{2}}(1-q)^{\frac{s-n+1}{2}}+(1-v_0)(1-q)^{\frac{s+n-1}{2}}q^{\frac{s-n+1}{2}}\right]\\
&+\frac{v_0q^{n+1}(1-q)+(1-v_0)q(1-q)^{n+1}}{v_0q^{n+1}+(1-v_0)(1-q)^{n+1}}\binom{s}{\frac{s+n+1}{2}}\left[v_0q^{\frac{s+n+1}{2}}(1-q)^{\frac{s-n-1}{2}}+(1-v_0)(1-q)^{\frac{s+n+1}{2}}q^{\frac{s-n-1}{2}}\right]\\
=&\left[\binom{s}{\frac{s+n-1}{2}}+\binom{s}{\frac{s+n+1}{2}}\right]\left[v_0q^{\frac{s+n+1}{2}}(1-q)^{\frac{s-n+1}{2}}+(1-v_0)(1-q)^{\frac{s-n+1}{2}}q^{\frac{s+n+1}{2}}\right]\\
=&\binom{s+1}{\frac{s+1+n}{2}}\left[v_0q^{\frac{s+1+n}{2}}(1-q)^{\frac{s+1-n}{2}}+(1-v_0)(1-q)^{\frac{s+1+n}{2}}q^{\frac{s+1-n}{2}}\right]=\zeta(\ell_{t+1}^r,n).
\end{aligned}$$

For $n=s+1$, we have $z_t(n+1)=0$ in (C.1) by (N) and hence

$$z_{t+1}(n)=\frac{v_0q^n+(1-v_0)(1-q)^n}{v_0q^{n-1}+(1-v_0)(1-q)^{n-1}}\left[v_0q^{n-1}+(1-v_0)(1-q)^{n-1}\right]$$

$$=[v_0 q^n + (1-v_0)(1-q)^n] = \zeta(\ell^r_{t+1}, n).$$

For $n=-s-1$, we have $z_t(n-1)=0$ in (C.1) by (N) and hence

$$\begin{aligned} z_{t+1}(n) =& \frac{v_0 q^{n+1}(1-q) + (1-v_0)q(1-q)^{n+1}}{v_0 q^{n+1} + (1-v_0)(1-q)^{n+1}} \left[v_0(1-q)^{-n-1} + (1-v_0)q^{-n-1}\right] \\ =& v_0 q^{n+1}(1-q) + (1-v_0)q(1-q)^{n+1}\left[q^{-n-1}(1-q)^{-n-1}\right] \\ =& \left[v_0(1-q)^{-n} + (1-v_0)q^{-n}\right] = \zeta(\ell^r_{t+1}, n). \end{aligned}$$

Thus, (12) holds for customer $t+1$. If $r^0_t(n) \equiv 0$ or equivalently $\ell^r_{t+1} = \ell^r_t = s$ on the other hand, we have $z_{t+1}(n) = z_t(n)\ \forall_n$ by (N) as $r^0_t(n) = 0\ \forall n$. Thus, our induction hypothesis immediately implies that (12) holds again for customer $t+1$. This completes the proof. ■

LEMMA C.1 **(Properties of $\zeta(\cdot,\cdot)$).** *For any integers $s \geq 0$ and $k \equiv s (mod\ 2)$, $\zeta(\cdot,\cdot)$ defined in (12) satisfies the following properties,*

$$\sum_n v_n \zeta(s,n) = \sum_n v_n \zeta(s+1,n) = v_0, \tag{C.2}$$

$$\sum_{n>k}(v_n - v^{**})\zeta(s,n) = \sum_{n>k+1}(v_n - v^{**})\zeta(s+1,n) + (v_{k+1} - v^{**})\zeta(s,k+2)(1-u_{k+2}), \tag{C.3}$$

$$\sum_{n>k-1}(v_n - v^{**})\zeta(s+1,n) = \sum_{n>k}(v_n - v^{**})\zeta(s,n) + (v_{k+1} - v^{**})\zeta(s,k)u_k, \tag{C.4}$$

$$\sum_{n<k}(v_n - v^{*})\zeta(s,n) = \sum_{n<k-1}(v_n - v^{*})\zeta(s+1,n) + (v_{k-1} - v^{*})\zeta(s,k-2)u_{k-2}, \tag{C.5}$$

$$\sum_{n<k+1}(v_n - v^{*})\zeta(s+1,n) = \sum_{n<k}(v_n - v^{*})\zeta(s,n) + (v_{k-1} - v^{*})\zeta(s,k)(1-u_k). \tag{C.6}$$

Proof of Lemma C.1. First, notice that

$$v_n = v_{n+1}u_n + v_{n-1}(1-u_n) \quad \forall n. \tag{C.7}$$

which is directly established by utilizing (8) and the description of u_n in Proposition 4:

$$\begin{aligned} v_n =& q v_n + (1-q)v_n \\ =& \underbrace{\frac{qv_0 + (1-q)(1-v_0)(\frac{1-q}{q})^n}{v_0 + (1-v_0)(\frac{1-q}{q})^{n+1}}}_{q} \underbrace{\frac{v_0}{v_0 + (1-v_0)(\frac{1-q}{q})^n}}_{v_n} + \underbrace{\frac{(1-q)v_0 + q(1-v_0)(\frac{1-q}{q})^n}{v_0 + (1-v_0)(\frac{1-q}{q})^{n-1}}}_{1-q} \underbrace{\frac{v_0}{v_0 + (1-v_0)(\frac{1-q}{q})^n}}_{v_n} \\ =& \underbrace{\frac{v_0}{v_0 + (1-v_0)(\frac{1-q}{q})^{n+1}}}_{v_{n+1}} \underbrace{\frac{qv_0 + (1-q)(1-v_0)(\frac{1-q}{q})^n}{v_0 + (1-v_0)(\frac{1-q}{q})^n}}_{u_n} + \underbrace{\frac{v_0}{v_0 + (1-v_0)(\frac{1-q}{q})^{n-1}}}_{v_{n-1}} \underbrace{\frac{(1-q)v_0 + q(1-v_0)(\frac{1-q}{q})^n}{v_0 + (1-v_0)(\frac{1-q}{q})^n}}_{(1-u_n)}. \end{aligned}$$

Now note that by Proposition 6, we can write (C.1) as

$$\zeta(s+1,n) = \zeta(s,n-1)u_{n-1} + \zeta(s,n+1)(1-u_{n+1}) \quad \forall_{n,s\geq 0}, \tag{C.8}$$

which, in turn, implies

$$\sum_n \zeta(s,n) = \sum_n \zeta(s+1,n) = 1. \quad \forall_{s\geq 0} \tag{C.9}$$

Where the second equality follows by the fact that $\zeta(0,0)=1$ and $\zeta(0,n)=0\ \forall_{n\neq 0}$. For any $s\geq 0$, by (C.7),

$$\begin{aligned}\sum_n v_n\zeta(s,n) &= \sum_n \{v_{n+1}\zeta(s,n)u_n + v_{n-1}\zeta(s,n)(1-u_n)\}\\ &= \sum_n \{v_n\zeta(s,n-1)u_{n-1} + v_n\zeta(s,n+1)(1-u_{n+1})\}\\ \text{by (C.8)} &= \sum_n v_n\zeta(s+1,n),\end{aligned}$$

hence establishing the first equality of (C.2). The second equality of (C.2) follows from the fact that $\zeta(0,0)=1$ and $\zeta(0,n)=0\ \forall n\neq 0$.

Notice that for an arbitrary constant c, by (C.2),

$$\sum_n (v_n-c)\zeta(s,n) = \sum_n (v_n-c)\zeta(s+1,n),$$

following from (C.9). Subtracting $\sum_{n\leq k}(v_n-c)\zeta(s,n)$ from both sides yields

$$\sum_{n>k}(v_n-c)\zeta(s,n) = \underbrace{\sum_n (v_n-c)\zeta(s+1,n)}_{A} - \underbrace{\sum_{n\leq k}(v_{n+1}-c)\zeta(s,n)u_n + (v_{n-1}-c)\zeta(s,n)(1-u_n)}_{B},$$

where B follows by (C.7). Then, re-arranging terms in A and B and utilizing (C.8) yield

$$\begin{aligned}A &= \sum_{n>k}(v_n-c)\zeta(s+1,n) + \sum_{n\leq k}(v_n-c)\zeta(s,n+1)(1-u_{n+1}) + \sum_{n\leq k}(v_n-c)\zeta(s,n-1)u_{n-1}, \quad \text{and}\\ B &= \sum_{n\leq k}(v_{n+2}-c)\zeta(s,n+1)u_{n+1} + \sum_{n\leq k}(v_n-c)\zeta(s,n+1)(1-u_{n+1}),\end{aligned}$$

which imply

$$\begin{aligned}\sum_{n>k}(v_n-c)\zeta(s,n) &= \sum_{n>k}(v_n-c)\zeta(s+1,n) + \sum_{n\leq k}(v_n-c)\zeta(s,n-1)u_{n-1} - \sum_{n\leq k}(v_{n+2}-c)\zeta(s,n+1)u_{n+1}\\ &= \sum_{n>k}(v_n-c)\zeta(s+1,n) - (v_{k+1}-c)\zeta(s,k)u_k\\ &= \sum_{n>k+1}(v_n-c)\zeta(s+1,n) + (v_{k+1}-c)[\underbrace{\zeta(s+1,k+1)-\zeta(s,k)u_k}_{=\zeta(s,k+2)(1-u_{k+2}) \text{ by (C.8)}}],\end{aligned}$$

establishing (C.3). Then, (C.4) follows immediately from (C.3) by noting that

$$\begin{aligned}\sum_{n>k}(v_n-c)\zeta(s,n) &= \sum_{n>k-1}(v_n-c)\zeta(s+1,n) - (v_{k+1}-c)\zeta(s+1,k+1) + (v_{k+1}-c)\zeta(s,k+2)(1-u_{k+2})\\ &= \sum_{n>k-1}(v_n-c)\zeta(s+1,n) - (v_{k+1}-c)\zeta(s,k)u_k,\end{aligned}$$

where the last equality again follows from (C.8).

To see (C.5), we have

$$\begin{aligned}\sum_{n<k}(v_n-c)\zeta(s,n) &= \sum_n (v_n-c)\zeta(s,n) - \sum_{n>k}(v_n-c)\zeta(s,n) - (v_k-c)\zeta(s,k)\\ \text{by (C.3)} \quad &= \sum_n (v_n-c)\zeta(s+1,n) - \sum_{n>k+1}(v_n-c)\zeta(s+1,n) - (v_{k+1}-c)\zeta(s,k+2)(1-u_{k+2}) - (v_k-c)\zeta(s,k)\\ &= \sum_{n<k-1}(v_n-c)\zeta(s+1,n) + (v_{k-1}-c)\qquad \underbrace{\zeta(s+1,k-1)}_{=\zeta(s,k-2)u_{k-2}+\zeta(s,k)(1-u_k) \text{ by (C.8)}}\end{aligned}$$

$$
\begin{aligned}
&+(v_{k+1}-c)[\underbrace{\zeta(s+1,k+1)-(v_{k+1}-c)\zeta(s,k+2)(1-u_{k+2})}_{=\zeta(s,k)u_k \text{ by (C.8)}}]-(v_k-c)\zeta(s,k)\\
=&\sum_{n<k-1}(v_n-c)\zeta(s+1,n)+(v_{k-1}-c)\zeta(s,k-2)u_{k-2}\\
&+\zeta(s,k)[\underbrace{(v_{k-1}-c)(1-u_k)+(v_{k+1}-c)u_k-(v_k-c)}_{=0 \text{ by (C.7)}}]\\
=&\sum_{n<k-1}(v_n-c)\zeta(s+1,n)+(v_{k-1}-c)\zeta(s,k-2)u_{k-2}.
\end{aligned}
$$

Finally, we show (C.6). By (C.5),

$$
\begin{aligned}
\sum_{n<k}(v_n-c)\zeta(s,n)&=\sum_{n<k-1}(v_n-c)\zeta(s+1,n)+(v_{k-1}-c)\zeta(s,k-2)u_{k-2}\\
\text{by (C.8)}&=\sum_{n<k-1}(v_n-c)\zeta(s+1,n)+(v_{k-1}-c)\zeta(s+1,k-1)-(v_{k-1}-c)\zeta(s,k)(1-u_k)\\
&=\sum_{n<k+1}(v_n-c)\zeta(s+1,n)-(v_{k-1}-c)\zeta(s,k)(1-u_k),
\end{aligned}
$$

establishing (C.6). ■

Proof of Proposition 7. First, note that when $r_t^0(n)=1$ for any n, (IC_0^r) can be written as $\sum_n v_n z_t(n)\in[v^*,v^{**})$. By Proposition 6, we can write the left-hand-side of the above constraint as

$$\sum_n v_n z_t(n)=\sum_n v_n\zeta(\ell_t^r,n)=v_0, \tag{C.10}$$

where the second equality follows from (C.2). Thus, (IC_0^r) is equivalent to $v_0\in[v^*,v^{**})$. Next, utilizing (3), we show that the expected revenue the platforms earns from neutral customer t is

$$
\begin{aligned}
p\,\mathbb{E}[\mathbb{1}[a_t=1]\mid m_t=0,r]&=p\,\mathbb{P}[a_t=1\mid m_t=0,r]\\
&=p\sum_n\underbrace{\mathbb{P}[S_t=1|N(H_t)=n]}_{=qv_n+(1-q)(1-v_n)}\underbrace{\mathbb{P}[N(H_t)=n|r]}_{z_t(n)}\\
&=p\,[q\underbrace{\sum_n v_n z_t(n)}_{=v_0 \text{ by (C.10)}}+(1-q)\underbrace{\sum_n v_n-v_n z_t(n))}_{=1-v_0 \text{ by (C.10) and (C.9)}}]=pu_0,
\end{aligned}
$$

following from $u_0=qv_0+(1-q)(1-v_0)$ definition given in Proposition 6. ■

LEMMA C.2. *If it is incentive compatible to partition customer t as an affirmative customer under an NA-partition policy $\hat r$ with $\ell_t^{\hat r}=s$, then there must uniquely exist $x\in[0,1)$ and $k\equiv s(\mathrm{mod}2)$ with $k\le s$, such that $\sum_n z_t(n)r_t^1(n)=\sum_n z_t(n)\hat r_t^1(n)$, $\sum_n(v_n-v^{**})z_t(n)r_t^1(n)\ge 0$ and $\sum_n(v_n-v^*)z_t(n)r_t^{-1}(n)\le 0$, where r is a threshold affirmative policy defined as*

$$r_t^1(n):=\begin{cases}1, & \text{for } n>k,\\ x\in[0,1), & \text{for } n=k,\\ 0, & \text{for } n<k,\end{cases}\quad \text{and}\quad r_t^{-1}(n):=1-r_t^1(n)\quad\forall n. \tag{C.11}$$

Proof of Lemma C.2. First we show the existence of k and x. If there do not exist $\underline{n}<\overline{n}$ with $\underline{n},\overline{n}\equiv s(\mathrm{mod}2)$ and $|\underline{n}|,|\overline{n}|\le s$ such that $\hat r_t^1(\underline{n})>0$ and $\hat r_t^{-1}(\overline{n})=1-\hat r_t^1(\overline{n})>0$, then there must exist $x\in[0,1)$ and $k\equiv s(\mathrm{mod}2)$ with $k\le s$ such that

$$\hat r_t^1(n)=\begin{cases}1, & \text{for } n>k,\\ x\in[0,1), & \text{for } n=k,\\ 0, & \text{for } n<k,\end{cases}\quad \text{and}\quad \hat r_t^{-1}(n)=1-\hat r_t^1(n)\quad\forall n\equiv s(\mathrm{mod}2)\text{ and }|n|\le s.$$

In this case, define r_t according to (C.11) with x and k identified above. Then, because $z_t(n)>0$ if and only if $n\equiv s(\mathrm{mod}2)$ and $|n|\le s$ by (12), we immediately have

$$\begin{aligned}\sum_n z_t(n)r_t^1(n)&=\sum_n z_t(n)\hat r_t^1(n),\\ \sum_n(v_n-v^{**})z_t(n)r_t^1(n)&=\sum_n(v_n-v^{**})z_t(n)\hat r_t^1(n)\ge 0,\ \text{and}\\ \sum_n(v_n-v^{*})z_t(n)r_t^{-1}(n)&=\sum_n(v_n-v^{*})z_t(n)\hat r_t^{-1}(n)\le 0.\end{aligned}$$

Now suppose that there exist $\underline{n}<\overline{n}$ with $\underline{n},\overline{n}\equiv s(\mathrm{mod}2)$ and $|\underline{n}|,|\overline{n}|\le s$ such that $\hat r_t^1(\underline{n})>0$ and $\hat r_t^{-1}(\overline{n})>0$. We demonstrate that $\hat r_t$ can be converted to a threshold affirmative policy $\tilde r_t$ with the desired properties (i.e., $\sum_n z_t(n)\tilde r_t^1(n)=\sum_n z_t(n)\hat r_t^1(n)$, $\sum_n(v_n-v^{**})z_t(n)\tilde r_t^1(n)\ge 0$ and $\sum_n(v_n-v^*)z_t(n)\tilde r_t^{-1}(n)\le 0$), in which such $\underline{n}$ and $\overline{n}$ do not exist. To that end, we define $\tilde r_t$ according to

$$\tilde r_t^1(\underline{n})=\hat r_t^1(\underline{n})-\alpha,\quad \tilde r_t^{-1}(\underline{n})=\hat r_t^{-1}(\underline{n})+\alpha,\quad \tilde r_t^{-1}(\overline{n})=\hat r_t^{-1}(\overline{n})-\beta,\quad \tilde r_t^{1}(\overline{n})=\hat r_t^{1}(\overline{n})+\beta,\ \text{and}\ \tilde r_t(n)=\hat r_t(n)\quad \forall n\ne\{\underline{n},\overline{n}\},$$

where $\alpha\in(0,\hat r_t^1(\underline{n})]$ and $\beta\in(0,\hat r_t^{-1}(\overline{n})]$ are given by

$$\alpha=\frac{z_t(\overline{n})}{z_t(\underline{n})}\hat r_t^{-1}(\overline{n})\ \text{and}\ \beta=\hat r_t^{-1}(\overline{n}),\ \text{if}\ \frac{\hat r_t^{-1}(\overline{n})}{\hat r_t^{1}(\underline{n})}\le\frac{z_t(\underline{n})}{z_t(\overline{n})},\tag{C.12}$$

$$\alpha=\hat r_t^{1}(\underline{n})\ \text{and}\ \beta=\frac{z_t(\underline{n})}{z_t(\overline{n})}\hat r_t^{1}(\underline{n}),\ \text{if}\ \frac{\hat r_t^{-1}(\overline{n})}{\hat r_t^{1}(\underline{n})}>\frac{z_t(\underline{n})}{z_t(\overline{n})}.\tag{C.13}$$

In both cases (C.12) and (C.13), we have $z_t(\underline{n})\alpha=z_t(\overline{n})\beta$. Thus,

$$\begin{aligned}\sum_n z_t(n)\tilde r_t^1(n)&=\sum_{n\ne\{\underline{n},\overline{n}\}} z_t(n)\tilde r_t^1(n)+z_t(\underline{n})\tilde r_t^1(\underline{n})+z_t(\overline{n})\tilde r_t^1(\overline{n})\\ &=\sum_{n\ne\{\underline{n},\overline{n}\}} z_t(n)\hat r_t^1(n)+z_t(\underline{n})\hat r_t^1(\underline{n})+z_t(\overline{n})\hat r_t^1(\overline{n})\underbrace{-z_t(\underline{n})\alpha+z_t(\overline{n})\beta}_{=0}\\ &=\sum_n z_t(n)\hat r_t^1(n),\end{aligned}\tag{C.14}$$

$$\begin{aligned}\sum_n(v_n-v^{**})z_t(n)\tilde r_t^1(n)&=\sum_n(v_n-v^{**})z_t(n)\hat r_t^1(n)+z_t(\overline{n})\beta(v_{\overline{n}}-v^{**})-z_t(\underline{n})\alpha(v_{\underline{n}}-v^{**}),\\ &=\sum_n(v_n-v^{**})z_t(n)\hat r_t^1(n)+z_t(\overline{n})\beta(v_{\overline{n}}-v^{**})-z_t(\overline{n})\beta(v_{\underline{n}}-v^{**}),\\ (\text{by (8), utilizing } \underline{n}<\overline{n})\quad &\ge\sum_n(v_n-v^{**})z_t(n)\hat r_t^1(n)\ge 0,\end{aligned}\tag{C.15}$$

$$\begin{aligned}\sum_n(v_n-v^{*})z_t(n)\tilde r_t^{-1}(n)&=\sum_n(v_n-v^{*})z_t(n)\hat r_t^{-1}(n)+z_t(\underline{n})\alpha(v_{\underline{n}}-v^{*})-z_t(\overline{n})\beta(v_{\overline{n}}-v^{*}),\\ &=\sum_n(v_n-v^{*})z_t(n)\hat r_t^{-1}(n)+z_t(\overline{n})\beta(v_{\underline{n}}-v^{*})-z_t(\overline{n})\beta(v_{\overline{n}}-v^{*}),\\ (\text{by (8), utilizing } \underline{n}<\overline{n})\quad &\le\sum_n(v_n-v^{*})z_t(n)\hat r_t^{-1}(n)\le 0.\end{aligned}\tag{C.16}$$

In the case of (C.12), we have $\tilde r_t^1(\overline{n})=1$ and hence $\tilde r_t^{-1}(\overline{n})=0$; in the case of (C.13), we have $\tilde r_t^{-1}(\underline{n})=1$ and hence $\tilde r_t^{1}(\underline{n})=0$. Thus, repeating the above iterative procedure until there do not exist $\underline{n}<\overline{n}$ with $\underline{n},\overline{n}\equiv s(\mathrm{mod}2)$ and $|\underline{n}|,|\overline{n}|\le s$ such that $\tilde r_t^1(\underline{n})>0$ and $\tilde r_t^{-1}(\overline{n})>0$ will result in a threshold affirmative policy $\tilde r_t$ (as in (C.11)) that preserves (C.14), (C.15) and (C.16).

To establish uniqueness of k and x, we first note that $\sum_n z_t(n)\hat{r}_t^1(n) \in [0,1)$. Suppose on the contrary, $\sum_n z_t(n)\hat{r}_t^1(n) = 1$, which is equivalent to $\hat{r}_t^1(n) = 1\ \forall n$ by (C.9). However, this contradicts with the fact that it is incentive compatible to partition customer t as an affirmative customer since (IC_1^r) would be violated, such that

$$\sum_n (v_n - v^{**}) z_t(n)\hat{r}_t^1(n) = \sum_n (v_n - v^{**}) z_t(n) \overset{\text{by (C.9)}}{=} \sum_n v_n z_t(n) - v^{**} \overset{\text{by (C.10)}}{=} v_0 - v^{**} < 0. \tag{C.17}$$

Next, notice by (C.9) (utilizing Proposition 6) that we have $\sum_{n>k} z_t(n) = 1$ if and only if $k < -s$ and $\sum_{n>k} z_t(n) = 0$ if and only if $k \geq s$. Therefore, as $\sum_{n>k} z_t(n)$ is non-increasing in k, there uniquely exist an integer $k \equiv s(\mathrm{mod}2)$ with $|k| \leq s$ such that $\sum_{n>k} z_t(n) \leq \sum_n z_t(n)\hat{r}_t^1(n) < \sum_{n>k-2} z_t(n)$. Hence $x = \frac{\sum_n z_t(n)\hat{r}_t^1(n) - \sum_{n>k} z_t(n)}{z_t(k)} \in [0,1)$. ■

Proof of Proposition 8. We divide the proof into three parts.

Part I. (14) **and** (15) **are well-defined,** which consists in showing the existence of $n^{**}(s), x^{**}(s), n^*(s)$ and $x^*(s)$ for any integer $s \geq 0$, as characterized by the following two claims.

Claim C.1. *$n^{**}(s) \equiv s(mod2), v_{n^{**}(s)} < v^{**}$ and $x^{**}(s) < 1$.*

Proof of Claim C.1 First, note that for any $n \not\equiv s(mod2)$ that satisfies the equality inside (14), we also have

$$\sum_{m>n-1} (v_m - v^{**})\zeta(s,m) = (v_n - v^{**})\zeta(s,n)x + \sum_{m>n} (v_m - v^{**})\zeta(s,m) = 0,$$

as $\zeta(s,n) = 0$ for all $n \not\equiv s(mod2)$ by (12). Hence, by (14), $n^{**}(s) \equiv s(mod2)$. Notice also that we must have $x^{**}(s) < 1$ if $n^{**}(s) = -s$, as otherwise, by (C.17) and the fact that $v_0 \in [v^*, v^{**})$, the equation inside (14) must be violated

$$\sum_{m\geq -s} (v_m - v^{**})\zeta(s,m) = \sum_m (v_m - v^{**})\zeta(s,m) = \sum_m (v_m)\zeta(s,m) - v^{**} = v_0 - v^{**} < 0.$$

Further, for any $n \geq -s+2$ that satisfies the equality inside (14) for $x = 1$, we also have

$$\sum_{m>n-2} (v_m - v^{**})\zeta(s,m) = (v_n - v^{**})\zeta(s,n) + \sum_{m>n} (v_m - v^{**})\zeta(s,m) = 0,$$

hence, by (14), (i) $x^{**}(s) < 1$.

- <u>When $v_s < v^{**}$,</u> we have $v_m < v^{**}$ for all m with $\zeta(s,m) > 0$ by (8). Hence, it is straightforward to verify that the equality inside (14) holds for $n = s, x = 0$. For any $n < s$, we have

$$\underbrace{(v_n - v^{**})}_{<0 \text{ as } v_n < v_s < v^{**}} \zeta(s,n)x + \sum_{m>n} \underbrace{(v_m - v^{**})\zeta(s,m)}_{<0 \text{ for } m \text{ with } \zeta(s,m)>0} \leq (v_s - v^{**})\zeta(s,s) < 0,$$

contradicting the equality inside (14). Notice that $v_{n^{**}(s)} < v^{**}$ as $n^{**}(s) = s$, by (8) and the fact that $v_s < v^{**}$.

- <u>When $v_s = v^{**}$,</u> we have $v_m \leq v^{**}$ for all m with $\zeta(s,m) > 0$ by (8). Hence, it is straightforward to verify that the equality inside (14) holds for $n = s-2, x = 0$. For any $n < s-2$, we have

$$\underbrace{(v_n - v^{**})}_{<0 \text{ as } v_n < v_s = v^{**}} \zeta(s,n)x + \sum_{m>n} \underbrace{(v_m - v^{**})\zeta(s,m)}_{\substack{<0 \text{ for } m<s \text{ with } \zeta(s,m)>0 \\ =0 \text{ for } m=s}} \leq (v_{s-2} - v^{**})\zeta(s,s-2) < 0,$$

contradicting the equality inside (14). Notice that $v_{n^{**}(s)} < v^{**}$ as $n^{**}(s) = s-2$, by (8) and the fact that $v_s = v^{**}$.

- When $v_s > v^{**}$, denote $f_s(n) := \sum_{m>n}(v_m - v^{**})\zeta(s,m)$. It is then straightforward to verify that $f_s(n)$ is positive and non-increasing in $n \leq s-2$ for $v_n \geq v^{**}$ (because $f_s(n) \geq f_s(s-2) = (v_s - v^{**})\zeta(s,s) > 0 = f_s(s)$), and $f_s(n)$ is non-decreasing in n for $v_n \leq v^{**}$ with $f_s(-s-2) = \sum_{m>-s-2}(v_m - v^{**})\zeta(s,m) = v_0 - v^{**} < 0$. Therefore, (ii) there exists a unique $n^{**}(s) \in [-s, s-2]$ satisfying $f_s(n^{**}(s)-2) < 0 \leq f_s(n^{**}(s))$. Accordingly, let

$$x^{**}(s) = \frac{\sum_{m>n^{**}(s)}(v_m - v^{**})\zeta(s,m)}{(v^{**} - v_{n^{**}(s)})\zeta(s, n^{**}(s))} = \frac{f_s(n^{**}(s)) - 0}{f_s(n^{**}(s)) - f_s(n^{**}(s)-2)} \in [0,1).$$

Then, it immediately follows that (iii) $(n^{**}(s), x^{**}(s))$ satisfies the equality inside (14). For any $n < n^{**}(s)$, the equation inside (14) must be violated, because then we must have (iv) $(v_n - v^{**}) < (v_{n^{**}(s)} - v^{**}) < 0$ by (8), as $(v^{**} - v_{n^{**}(s)}) \geq (v^{**} - v_{n^{**}(s)})\zeta(s, n^{**}(s)) = f_s(n^{**}(s)) - f_s(n^{**}(s)-2) > 0$ by (ii). Then, the equation inside (14) can be written as

$$\begin{aligned} 0 = &\underbrace{(v_n - v^{**})\zeta(s,n)}_{\leq 0 \text{ by (iv)}} x + \underbrace{(v_{n^{**}(s)} - v^{**})\zeta(s, n^{**}(s))}_{<0 \text{ by (ii)}} \underbrace{(1 - x^{**}(s))}_{>0 \text{ by (i)}} \\ &+ \underbrace{(v_{n^{**}(s)} - v^{**})\zeta(s, n^{**}(s))x^{**}(s) + \sum_{m>n^{**}(s)}(v_m - v^{**})\zeta(s,m)}_{=0 \text{ by (iii)}} + \sum_{m\in(n, n^{**}(s))}(v_m - v^{**})\zeta(s,m), \end{aligned}$$

leading to a contradiction, whereby we use the fact that $v_n < v_m < v_{n^{**}(s)} < v^{**}$ for all (if any) $m \in (n, n^{**}(s))$ by (8). Lastly, notice that $v_{n^{**}(s)} < v^{**}$ as $v_s > v^{**}$ and $\zeta(s, n^{**}(s)) > 0$ by (ii). $\square$

Claim C.2. *$n^*(s) \equiv s(mod2), v_{n^*(s)} \geq v^*$ and $x^*(s) \leq 1$, where the second equality holds only when $v_0 = v^*$ and $s = 0$, and the third equality holds only when $v_0 = v^*$.*

Proof of Claim C.2 First, note that for any $n \not\equiv s(mod2)$ that satisfies the equality inside (15), we also have

$$\sum_{m<n+1}(v_m - v^*)\zeta(s,m) = (v_n - v^*)\zeta(s,n)(1-x) + \sum_{m<n}(v_m - v^*)\zeta(s,m) = 0,$$

as $\zeta(s,n) = 0$ for all $n \not\equiv s(mod2)$ by (12). Hence, by (15), $n^{**}(s) \equiv s(mod2)$. Next, notice that for any $n \leq s-2$ that satisfies the equality inside (15) for $x = 0$, we also have

$$\sum_{m<n+2}(v_m - v^*)\zeta(s,m) = (v_n - v^*)\zeta(s,n) + \sum_{m<n}(v_m - v^*)\zeta(s,m) = 0,$$

hence, by (15), (v) $x^*(s) > 0$.

- When $v_{-s} > v^*$, we have $v_m > v^*$ for all m with $\zeta(s,m) > 0$ by (8). Hence, it is straightforward to verify that the equality inside (15) holds for $n = -s, x = 1$. For any $n > -s$, we have

$$\underbrace{(v_n - v^*)}_{>0 \text{ as } v_n > v_{-s} > v^*} \underbrace{\zeta(s,n)}_{\substack{>0 \text{ as} \\ n \equiv s(mod2), n \leq s}} (1-x) + \sum_{m<n} \underbrace{(v_m - v^*)\zeta(s,m)}_{>0 \text{ for } m \text{ with } \zeta(s,m)>0} \geq (v_{-s} - v^*)\zeta(s,s) > 0,$$

contradicting the equality inside (15). Notice that $v_{n^*(s)} > v^*$ as $n^*(s) = -s$, by (8) and the fact that $v_{-s} > v^*$.

- When $v_{-s} = v^*$, we have $v_m \geq v^*$ for all m with $\zeta(s,m) > 0$ by (8). Hence, it is straightforward to verify that the equality inside (15) holds for $n = -s+2, x = 0$. For any $n > -s+2$, we have

$$\underbrace{(v_n - v^*)}_{\substack{>0 \text{ as} \\ v_n > v_{-s} = v^*}} \underbrace{\zeta(s,n)}_{\substack{>0 \text{ as} \\ n \equiv s(mod2), n \leq s}} x + \sum_{m<n} \underbrace{(v_m - v^*)\zeta(s,m)}_{\substack{>0 \text{ for } m>-s \text{ with } \zeta(s,m)>0 \\ =0 \text{ for } m=-s}} \geq (v_{-s+2} - v^*)\zeta(s, -s+2) > 0,$$

contradicting the equality inside (15). Notice that $v_{n^*(s)} > v^*$ by (8) and the fact that $v_{-s} = v^*$.

- When $v_{-s} < v^*$, denote $f_s(n) := \sum_{m<n}(v_m - v^*)\zeta(s,m)$. When $v_{-s} < v^*$, it is straightforward to verify that $f_s(n)$ is negative and non-increasing in $n \geq -s+2$ for $v_n \leq v^*$ (because $f_s(n) \leq f_s(-s+2) = (v_{-s} - v^*)\zeta(s,-s) < 0 = f_s(-s)$), and $f_s(n)$ is non-decreasing in n for $v_n \geq v^*$ with $f_s(s+2) = \sum_{m<s+2}(v_m - v^*)\zeta(s,m) = v_0 - v^* \geq 0$. Therefore, (vi) there exists a unique $n^*(s) \in [-s+2, s]$ satisfying $f_s(n^*(s)) \leq 0 < f_s(n^*(s)+2)$. Accordingly, let

$$1 - x^*(s) = \frac{\sum_{m<n^*(s)}(v_m - v^*)\zeta(s,m)}{(v^* - v_{n^*(s)})\zeta(s,n^*(s))} = \frac{f_s(n^*(s)) - 0}{f_s(n^*(s)) - f_s(n^*(s)+2)} \in [0,1).$$

Then, it immediately follows that (vii) $(n^*(s), x^*(s))$ satisfies the equality inside (15). For any $n > n^*(s)$, the equation inside (15) must be violated, because then we must have (viii) $(v_n - v^*) > (v_{n^*(s)} - v^*) > 0$ by (8), as $(v_{n^*(s)} - v^*) \geq (v_{n^*(s)} - v^*)\zeta(s,n^*(s)) = f_s(n^*(s)+2) - f_s(n^*(s)) > 0$ by (vi). Then, the equation inside (15) can be written as

$$\begin{aligned} 0 = &\underbrace{(v_n - v^*)}_{>0 \text{ by (viii)}} \underbrace{\zeta(s,n)}_{\substack{>0 \text{ by (N)} \\ \text{as } n \leq s}} (1-x) + \underbrace{(v_{n^*(s)} - v^*)\zeta(s,n^*(s))}_{<0 \text{ by (vi)}} \underbrace{(1-x^*(s))}_{>0 \text{ by (v)}} \\ &+ \underbrace{(v_{n^*(s)} - v^*)\zeta(s,n^*(s))(1-x^*(s)) + \sum_{m<n^*(s)}(v_m - v^*)\zeta(s,m)}_{=0 \text{ by (vii)}} + \sum_{m\in(n^*(s),n)}(v_m - v^*)\zeta(s,m), \end{aligned}$$

leading to a contradiction, whereby we use the fact that $v^* < v_{n^*(s)} < v_m < v_n$ for all (if any) $m \in (n^*(s), n)$ by (8). Lastly, notice that $v_{n^*(s)} > v^*$ as $v_{-s} < v^*$ and $\zeta(s,n^*(s)) > 0$ by (vi). $\square$

Part II. It is incentive compatible to partition customer t as an affirmative customer if and only if $n^*(s) > n^{**}(s)$ or $n^*(s) = n^{**}(s)$ with $x^{**}(s) \geq x^*(s)$, where $s = \ell_t^r$.

To show the sufficiency, we now demonstrate the following threshold affirmative policy simultaneously satisfies (IC_1^r) and (IC_{-1}^r):

$$r_t^1(n) := \begin{cases} 1, & \text{for } n > n^{**}(s), \\ x^{**}(s) \in [0,1], & \text{for } n = n^{**}(s), \\ 0, & \text{for } n < n^{**}(s), \end{cases} \quad \text{and} \quad r_t^{-1}(n) := 1 - r_t^1(n) \quad \forall n. \tag{C.18}$$

Indeed, by (14), we have

$$\sum_n (v_n - v^{**}) z_t(n) r_t^1(n) = \left(v_{n^{**}(s)} - v^{**}\right)\zeta(s,n^{**}(s))x^{**}(s) + \sum_{n>n^{**}(s)}(v_n - v^{**})\zeta(s,n) = 0, \tag{C.19}$$

establishing (IC_1^r). To establish (IC_{-1}^r), we now show

$$\sum_n (v_n - v^*) z_t(n) r_t^{-1}(n) = \left(v_{n^{**}(s)} - v^*\right)\zeta(s,n^{**}(s))(1 - x^{**}(s)) + \sum_{n<n^{**}(s)}(v_n - v^*)\zeta(s,n) \leq 0. \tag{C.20}$$

- For $n^*(s) > n^{**}(s)$, by Claim C.2 we have $v_{n^*(s)} \geq v^*$, which leads to $(v_{n^*(s)} - v^*)\zeta(s,n^*(s))(1-x^*(s)) \geq 0$. Hence, by (15), we have $\sum_{n<n^*(s)}(v_n - v^*)\zeta(s,n) \leq 0$. Then, it is straightforward to see that (C.20) holds, such that

$$\begin{aligned} 0 \geq \sum_{n<n^*(s)}(v_n - v^*)\zeta(s,n) &\geq \max\left\{\sum_{n<n^{**}(s)}(v_n - v^*)\zeta(s,n), \sum_{n\leq n^{**}(s)}(v_n - v^*)\zeta(s,n)\right\} \\ &\geq \left(v_{n^{**}(s)} - v^*\right)\zeta(s,n^{**}(s))(1 - x^{**}(s)) + \sum_{n<n^{**}(s)}(v_n - v^*)\zeta(s,n), \end{aligned}$$

where the second inequality follows by $n^*(s) > n^{**}(s)$ as v_n increases in n by (8), and the third inequality follows from the fact that $x^{**}(s) \in [0,1]$.

- For $n^*(s) = n^{**}(s)$ with $x^{**}(s) \geq x^*(s)$, by Claim C.2 we have $v_{n^*(s)} \geq v^*$, which leads to $(v_{n^*(s)} - v^*)\zeta(s, n^*(s))(1 - x^*(s)) \geq 0$. Hence, by (15), we have

$$
\begin{aligned}
0 &= \underbrace{\left(v_{n^*(s)} - v^*\right)\zeta(s, n^*(s))(1 - x^*(s))}_{\geq 0} + \sum_{n < n^*(s)} (v_n - v^*)\zeta(s,n) \\
&= \left(v_{n^{**}(s)} - v^*\right)\zeta(s, n^{**}(s))(1 - x^*(s)) + \sum_{n < n^{**}(s)} (v_n - v^*)\zeta(s,n) \\
&\geq \left(v_{n^{**}(s)} - v^*\right)\zeta(s, n^{**}(s))(1 - x^{**}(s)) + \sum_{n < n^{**}(s)} (v_n - v^*)\zeta(s,n),
\end{aligned}
$$

where the second equality follows by $n^*(s) = n^{**}(s)$, and the third inequality follows from $v_{n^{**}(s)} = v_{n^*(s)} \geq v^*$ by Claim C.2 and the fact that $x^{**}(s) \geq x^*(s)$. Thus, (C.20) holds.

To show <u>necessity</u>, we note that, if it is incentive compatible to partition customer t (when $\ell_t^r = s$) as an affirmative customer, Lemma C.2 implies the existence of $x \in [0,1)$ and $k \equiv s(\text{mod}2)$ with $k \leq s$, such that r_t specified in (C.18) satisfies (IC_1^r),

$$
\sum_n (v_n - v^{**}) z_t(n) r_t^1(n) = (v_k - v^{**})\,\zeta(s,k)x + \sum_{n>k} (v_n - v^{**})\zeta(s,n) \geq 0, \tag{C.21}
$$

and (IC_{-1}^r),

$$
\sum_n (v_n - v^*) z_t(n) r_t^{-1}(n) = (v_k - v^*)\,\zeta(s,k)(1-x) + \sum_{n<k} (v_n - v^*)\zeta(s,n) \leq 0. \tag{C.22}
$$

We now claim (C.21) and (C.22) must imply $n^*(s) > n^{**}(s)$ or $n^*(s) = n^{**}(s)$ with $x^{**}(s) \geq x^*(s)$. To that end, we first establish the following lemma, whose proof is relegated after the proof of Proposition 8.

LEMMA C.3. *If it is incentive compatible to partition customer t (when $\ell_t^r = s$) as an affirmative customer, $(n^{**}(s), x^{**}(s))$, $(n^*(s), x^*(s))$ and (x,k) determined by (14), (15) and (C.18) of Lemma C.2 respectively, satisfy*

$$
\textit{either} \quad k > n^{**}(s) \quad \textit{or} \quad k = n^{**}(s) \ \textit{with}\ x \leq x^{**}(s), \quad \textit{and} \tag{C.23}
$$

$$
\textit{either} \quad k < n^*(s) \quad \textit{or} \quad k = n^*(s) \ \textit{with}\ x^*(s) \leq x. \tag{C.24}
$$

Lemma C.3 immediately implies that $n^*(s) \geq n^{**}(s)$, as $n^*(s) \geq k \geq n^{**}(s)$ by (C.23) and (C.24). In particular, when $n^*(s) = n^{**}(s) = k$, we have $x^{**}(s) \geq x \geq x^*(s)$.

Part III. If it is incentive compatible to partition customer t as an affirmative customer, then partitioning any other customer $t' > t$ as an affirmative customer will also be incentive compatible; and it is optimal for the platform to offer affirmative recommendations according to (C.18) and to earn an expected revenue of $pF(s)$, where

$$
F(s) = x^{**}(s)\zeta(s, n^{**}(s)) + \sum_{n > n^{**}(s)} \zeta(s,n),
$$

is a non-decreasing function in s. In Part II, we show for customer t with $\ell_t^r = s \geq \tau^\circ$, that the threshold affirmative policy given by (C.18) simultaneously satisfies (IC_1^r) and (IC_{-1}^r) for $n^*(s) > n^{**}(s)$ or $n^*(s) = n^{**}(s)$ with $x^{**}(s) \geq x^*(s)$. Now, for customer $t' > t$ with $\ell_{t'}^r = s+1$, consider the following policy

$$
r_{t'}^1(n) := \begin{cases} 1, & \text{for } n > n^{**}(s)+1, \\ \frac{x^{**}(s)\zeta(s,n^{**}(s))u_{n^{**}(s)} + \zeta(s,n^{**}(s)+2)(1-u_{n^{**}(s)+2})}{\zeta(s+1,n^{**}(s)+1)}, & \text{for } n = n^{**}(s)+1, \\ \frac{x^{**}(s)\zeta(s,n^{**}(s))(1-u_{n^{**}(s)})}{\zeta(s+1,n^{**}(s)-1)}, & \text{for } n = n^{**}(s)-1, \\ 0, & \text{for } n < n^{**}(s)-1, \end{cases} \quad \text{and} \quad r_{t'}^{-1}(n) := 1 - r_{t'}^1(n) \quad \forall n. \tag{C.25}
$$

Then, we have

$$\begin{aligned}
\sum_n (v_n - v^{**})\zeta(s+1,n) r^1_{t'}(n) &= \sum_{n>n^{**}(s)+1} (v_n - v^{**})\zeta(s+1,n) \\
&\quad + (v_{n^{**}(s)+1} - v^{**}) \left[x^{**}(s)\zeta(s,n^{**}(s))u_{n^{**}(s)} + \zeta(s,n^{**}(s)+2)(1-u_{n^{**}(s)+2})\right] \\
&\quad + (v_{n^{**}(s)-1} - v^{**}) \left[x^{**}(s)\zeta(s,n^{**}(s))(1-u_{n^{**}(s)})\right] \\
&= \underbrace{\sum_{n>n^{**}(s)+1} (v_n - v^{**})\zeta(s+1,n) + (v_{n^{**}(s)+1} - v^{**})\zeta(s,n^{**}(s)+2)(1-u_{n^{**}(s)+2})}_{\sum_{n>n^{**}(s)}(v_n-v^{**})\zeta(s,n) \text{ by (C.3)}} \\
&\quad + x^{**}(s)\zeta(s,n^{**}(s)) \underbrace{\left[(v_{n^{**}(s)+1} - v^{**})u_{n^{**}(s)} + (v_{n^{**}(s)-1} - v^{**})(1-u_{n^{**}(s)})\right]}_{v_{n^{**}(s)}-v^{**} \text{ by (C.7)}} \\
\text{(by (C.19))} \quad &= \left(v_{n^{**}(s)} - v^{**}\right)\zeta(s,n^{**}(s))x^{**}(s) + \sum_{n>n^{**}(s)} (v_n - v^{**})\zeta(s,n) = 0.
\end{aligned}$$

Therefore, the policy given in (C.25) satisfies (IC^r_1). Similarly, we also have

$$\begin{aligned}
\sum_n (v_n - v^{*})\zeta(s+1,n) r^{-1}_{t'}(n) &= \sum_{n<n^{**}(s)-1} (v_n - v^{*})\zeta(s+1,n) \\
&\quad + (v_{n^{**}(s)-1} - v^{*}) \left[(1-x^{**}(s))\zeta(s,n^{**}(s))(1-u_{n^{**}(s)}) + \zeta(s,n^{**}(s)-2)u_{n^{**}(s)-2}\right] \\
&\quad + (v_{n^{**}(s)+1} - v^{*}) \left[(1-x^{**}(s))\zeta(s,n^{**}(s))u_{n^{**}(s)}\right] \\
&= \underbrace{\sum_{n<n^{**}(s)-1} (v_n - v^{*})\zeta(s+1,n) + (v_{n^{**}(s)-1} - v^{*})\zeta(s,n^{**}(s)-2)u_{n^{**}(s)-2}}_{\sum_{n<n^{**}(s)}(v_n-v^{*})\zeta(s,n) \text{ by (C.5)}} \\
&\quad + (1-x^{**}(s))\zeta(s,n^{**}(s)) \underbrace{\left[(v_{n^{**}(s)-1} - v^{*})(1-u_{n^{**}(s)}) + (v_{n^{**}(s)+1} - v^{*})u_{n^{**}(s)}\right]}_{v_{n^{**}(s)}-v^{*} \text{ by (C.7)}} \\
\text{(by (C.20))} \quad &= \left(v_{n^{**}(s)} - v^{*}\right)\zeta(s,n^{**}(s))(1-x^{**}(s)) + \sum_{n<n^{**}(s)} (v_n - v^{*})\zeta(s,n) \le 0.
\end{aligned}$$

Therefore, the policy given in (C.25) also satisfies (IC^r_{-1}). Hence, in a recursive manner, we show that if it is incentive compatible to partition customer t as an affirmative customer, then partitioning any other customer $t' > t$ as an affirmative customer will also be incentive compatible. It is straightforward to see, by Lemma C.2 and (14), that offering affirmative recommendations according to (C.18) is optimal for the platform. The corresponding expected revenue is then calculated by (11), utilizing (12). Lastly, the expected revenue that policy given in (C.25) generated from customer t' is as follows

$$\sum_n r^1_{t'}(n)\zeta(s+1,n) = \underbrace{\sum_{n>n^{**}(s)+1} \zeta(s+1,n) + \zeta(s,n^{**}(s)+2)(1-u_{n^{**}(s)+2})}_{=\sum_{n>n^{**}(s)}\zeta(s,n) \text{ by (C.8)}} + x^{**}(s)\zeta(s,n^{**}(s)) = F(s),$$

which immediately implies $F(s) \le F(s+1)$ by Lemma C.2. ■

Proof of Lemma C.3. First, we establish (C.23), by showing

- $\underline{k \ge n^{**}(s)}$. Suppose $k < n^{**}(s)$, then (i) $v_k < v_{n^{**}(s)} < v^{**}$ by (8) (first inequality) and Claim C.1 (second inequality). We then have

$$(v_k - v^{**})\zeta(s,k)x + \sum_{m>k} (v_m - v^{**})\zeta(s,m)$$

$$
\begin{aligned}
&= \underbrace{(v_k - v^{**})\zeta(s,k)}_{\leq 0 \text{ by (i)}} x + \underbrace{(v_{n^{**}(s)} - v^{**})\zeta(s,n^{**}(s))}_{<0 \text{ by Claim C.1}} \underbrace{(1-x^{**}(s))}_{>0 \text{ by Claim C.1}} + \sum_{m\in(k,n^{**}(s))} (v_m - v^{**})\zeta(s,m) \\
&\quad \underbrace{+ (v_{n^{**}(s)} - v^{**})\zeta(s,n^{**}(s))x^{**}(s) + \sum_{m>n^{**}(s)} (v_m - v^{**})\zeta(s,m)}_{=0 \text{ by (14)}} < 0,
\end{aligned}
$$

leading to a contradiction with (C.21), whereby we use the fact that $v_k < v_m < v_{n^{**}(s)} < v^{**}$ for all (if any) $m \in (k, n^{**}(s))$ by (8).

- $\underline{x \leq x^{**}(s) \text{ when } k = n^{**}(s)}$. Suppose $x > x^{**}(s)$ when $k = n^{**}(s)$, we then have

$$
\begin{aligned}
&(v_k - v^{**})\zeta(s,k)x + \sum_{m>k}(v_m - v^{**})\zeta(s,m) \\
&\quad = (v_{n^{**}(s)} - v^{**})\zeta(s,n^{**}(s))x + \sum_{m>n^{**}(s)} (v_m - v^{**})\zeta(s,m) \\
&\quad = \underbrace{(v_{n^{**}(s)} - v^{**})\zeta(s,n^{**}(s))}_{<0 \text{ by Claim C.1}} \underbrace{(x - x^{**}(s))}_{>0} + \underbrace{(v_{n^{**}(s)} - v^{**})\zeta(s,n^{**}(s))x^{**}(s) + \sum_{m>n^{**}(s)} (v_m - v^{**})\zeta(s,m)}_{=0 \text{ by (14)}} \\
&\quad < 0,
\end{aligned}
$$

leading to a contradiction with (C.21).

Next, we establish (C.24), by showing

- $\underline{k \leq n^{*}(s)}$. First, note that (ii) $v_{n^*(s)+2} > v^*$ by (8) and Claim C.2. Further, we have (iii) $(v_{n^*(s)} - v^*)\zeta(s,n^*(s)) \geq 0$, again by Claim C.2. Suppose $k > n^*(s)$, we then have

$$
\begin{aligned}
(v_k - v^*)\zeta(s,k)(1-x) + \sum_{m<k}(v_m - v^*)\zeta(s,m) &= \underbrace{(v_{n^*(s)} - v^*)\zeta(s,n^*(s))(1-x^*(s)) + \sum_{m<n^*(s)} (v_m - v^*)\zeta(s,m)}_{=0 \text{ by (15)}} \\
&\quad + \underbrace{(v_{n^*(s)} - v^*)\zeta(s,n^*(s))x^*(s)}_{\geq 0 \text{ by (iii)}} + \underbrace{(v_k - v^*)\zeta(s,k)(1-x)}_{\substack{>0 \text{ by (8) and (ii) as } k\geq n^*(s)+2, \\ |k|\leq s \text{ by Lemma C.2 and } x<1}} \\
&\quad + \sum_{m\in(n^*(s),k)} (v_m - v^*)\zeta(s,m) \\
&> 0,
\end{aligned}
$$

leading to a contradiction with (C.22), whereby we use the fact that $v^* < v_{n^*(s)+2} \leq v_m$ for all (if any) $m \in (n^*(s), k)$ by (ii).

- $\underline{x^*(s) \leq x \text{ when } k = n^*(s)}$. Suppose $x^*(s) > x$ when $k = n^*(s)$. Then, as $x \in [0,1)$, we must have $x^*(s) > 0$, which leads to $v_{n^*(s)} > v^*$ by Claim C.2. Therefore, (iv) $(v_{n^*(s)} - v^*)\zeta(s,n^*(s)) > 0$. We then have

$$
\begin{aligned}
(v_k - v^*)\zeta(s,k)(1-x) + \sum_{m<k}(v_m - v^*)\zeta(s,m) &= (v_{n^*(s)} - v^*)\zeta(s,n^*(s))(1-x) + \sum_{m<n^*(s)} (v_m - v^*)\zeta(s,m) \\
&> \underbrace{(v_{n^*(s)} - v^*)\zeta(s,n^*(s))x^*(s) + \sum_{m<n^*(s)} (v_m - v^*)\zeta(s,m)}_{=0 \text{ by (15)}}
\end{aligned}
$$

leading to a contradiction with (C.22), where the second inequality follows by (iv) and the fact that $x^*(s) > x$. This completes the proof. ■

Proof of Proposition 9. Under an NA-partition policy, the selling horizon is essentially partitioned into either *neutral customers* $\mathcal{N}(r) := \{t = 1, \cdots, T : r_t^0(n) \equiv 1 \text{ for all } n\}$ or *affirmative customers* $\mathcal{A}(r) := \{t = 1, \cdots, T : r_t^0(n) \equiv 0 \text{ for all } n\}$. First, we note that under an NA-partition policy r, the platform earns an expected revenue of pu_0 for customer $t \in \mathcal{N}(r)$ and an optimal expected revenue of $pF(\ell_t^r)$ for customer $t \in \mathcal{A}(r)$. Therefore, the determination of the platform's optimal NA-partition policy reduces to the determination of the neutral and affirmative customers, i.e.,

$$p \max_{\mathcal{N}(r), \mathcal{A}(r)} \sum_{t \in \mathcal{N}(r)} u_0 + \sum_{t \in \mathcal{A}(r)} F(\ell_t^r). \tag{C.26}$$

Thus, to establish (17), it suffices to show that an NA-partition policy r with $t_1 \in \mathcal{A}(r)$ and $t_2 \in \mathcal{N}(r)$ for some $\tau^\circ \leq t_1 < t_2 \leq T$ yields less revenue than the modified NA-partition policy, $\tilde{r}$, where (i) $\mathcal{A}(\tilde{r}) \equiv \mathcal{A}(r) \setminus \{t_1\} \cup \{t_2\}$ and (ii) $\mathcal{N}(\tilde{r}) \equiv \mathcal{N}(r) \cup \{t_1\} \setminus \{t_2\}$. To that end, we utilize the platform's objective function representation in (C.26), such that

$$\begin{aligned}
\sum_{t \in \mathcal{N}(\tilde{r})} u_0 + \sum_{t \in \mathcal{A}(\tilde{r})} F(\ell_t^{\tilde{r}}) &= \sum_{t \in \mathcal{N}(\tilde{r})} u_0 + \sum_{t < t_1, t \in \mathcal{A}(\tilde{r})} F(\ell_t^{\tilde{r}}) + \sum_{t \in (t_1, t_2), t \in \mathcal{A}(\tilde{r})} F(\ell_t^{\tilde{r}}) + F(\ell_{t_2}^{\tilde{r}}) + \sum_{t > t_2, t \in \mathcal{A}(\tilde{r})} F(\ell_t^{\tilde{r}}) \\
&= \sum_{t \in \mathcal{N}(r)} u_0 + \sum_{t < t_1, t \in \mathcal{A}(r)} F(\ell_t^r) + \underbrace{\sum_{t \in (t_1, t_2), t \in \mathcal{A}(\tilde{r})} F(\ell_t^{\tilde{r}})}_{\sum_{t \in (t_1, t_2), t \in \mathcal{A}(r)} F(\ell_t^r + 1)} + \underbrace{F(\ell_{t_2}^{\tilde{r}})}_{F\left(\ell_{t_1}^r + \sum_t \mathbb{1}[t \in (t_1, t_2), t \in \mathcal{N}(r)]\right)} \\
&\quad + \sum_{t > t_2, t \in \mathcal{A}(r)} F(\ell_t^r) \\
&\geq \sum_{t \in \mathcal{N}(r)} u_0 + \sum_{t < t_1, t \in \mathcal{A}(r)} F(\ell_t^r) + F(\ell_{t_1}^r) + \sum_{t \in (t_1, t_2), t \in \mathcal{A}(r)} F(\ell_t^r) + \sum_{t > t_2, t \in \mathcal{A}(r)} F(\ell_t^r) \\
&= \sum_{t \in \mathcal{N}(r)} u_0 + \sum_{t \in \mathcal{A}(r)} F(\ell_t^r),
\end{aligned}$$

where the second equality follows by (i),(ii) and the fact that $\sum_{t \in [t_1, t_2]} \mathbb{1}[t \in \mathcal{N}(\tilde{r})] = \sum_{t \in [t_1, t_2]} \mathbb{1}[t \in \mathcal{N}(r)]$, and the inequality follows by the non-decreasing property of $F(s)$ in s by Proposition 8. It is straightforward to see (IC_1^r) and (IC_{-1}^r) are satisfied with $\tilde{r}$ as well, since $\tau^\circ \leq \min \mathcal{A}(r) \leq \min \mathcal{A}(\tilde{r})$ by (i),(ii) and the fact that r is a feasible NA-partition policy. ■

Proposition C.1 **(Full-disclosure policy)**. *The platform's expected revenue under the full-disclosure policy is given by*

$$\pi^{FD} = p \begin{cases}
\dfrac{u_0(Tu_1+1-u_0)-[u_0(1-u_1)]^{\frac{T}{2}}\left[T+Tu_0^2(u_1-1)+2u_0u_1(u_0-1)\right]}{1-u_0(1-u_1)} \\
\quad + \dfrac{2u_0(1-u_0)\left[(1-u_1)\left[u_0-[u_0(1-u_1)]^{\frac{T+2}{2}}\right]-u_1\right]}{[1-u_0(1-u_1)]^2}, & \text{for } v_0 \in [v^*, p) \text{ and even } T, \\
\dfrac{u_0(T+u_0-1)+u_0[(1-u_0)u_{-1}]^{\frac{T}{2}}[1-u_0(T+1)-(T+2)(1-u_0)(1-u_{-1})]}{1-(1-u_0)u_{-1}} \\
\quad + \dfrac{2u_0(1-u_0)\left[u_{-1}\left[[(1-u_0)u_{-1}]^{\frac{T+2}{2}}-1+u_0\right]+1-u_{-1}\right]}{[1-(1-u_0)u_{-1}]^2}, & \text{for } v_0 \in [p, v^{**}) \text{ and even } T, \\
\dfrac{u_0(Tu_1+1-u_0)-[u_0(1-u_1)]^{\frac{T+1}{2}}\left[(T+2)u_0(1-u_0)+[u_0(T+1)-1]\frac{u_1}{1-u_1}\right]}{1-u_0(1-u_1)} \\
\quad + \dfrac{2u_0(1-u_0)\left[1-[u_0(1-u_1)]^{\frac{T+1}{2}}\right][u_0(1-u_1)-u_1]}{[1-u_0(1-u_1)]^2}, & \text{for } v_0 \in [v^*, p) \text{ and odd } T, \\
\dfrac{u_0(T+u_0-1)+u_0[(1-u_0)u_{-1}]^{\frac{T+1}{2}}\left[1-u_0(T+2)-(T+1)\frac{1-u_{-1}}{u_{-1}}\right]}{1-(1-u_0)u_{-1}} \\
\quad + \dfrac{2u_0(1-u_0)\left[1-[(1-u_0)u_{-1}]^{\frac{T+1}{2}}\right][1-u_{-1}-(1-u_0)u_{-1}]}{[1-(1-u_0)u_{-1}]^2}, & \text{for } v_0 \in [p, v^{**}) \text{ and odd } T.
\end{cases} \tag{C.27}$$

In particular, π^{FD} can have discontinuity in v_0 or p only at $v_0 = p$.

Proof of Proposition C.1. We first establish the following two claims C.3 and C.4.

CLAIM C.3. *The full-disclosure policy induces the same purchase decisions from all customers and the same expected revenue for the platform as the following recommendation policy:*

$$(r_t^1(n), r_t^0(n), r_t^{-1}(n)) = \begin{cases} (1,0,0), & \text{if } n \geq n^+, \\ (0,1,0), & \text{if } n \in (n^-, n^+), \\ (0,0,1), & \text{if } n \leq n^-, \end{cases} \quad \text{for all } t \in \{1,2,...,T\}, \tag{C.28}$$

where $n^+ := \min_{n\in\mathbb{Z}} \{n : v_n \geq v^{**}\}$ and $n^- := \max_{n\in\mathbb{Z}} \{n : v_n < v^*\}$. Let $N(H_t)$ be the net purchase position up to time t under the recommendation policy in (C.28), and denote $\tau^+ := \min\{t = 1, \cdots, T : N(H_{t+1}) \geq n^+\}$ and $\tau^- := \min\{t = 1, \cdots, T : N(H_{t+1}) \leq n^-\}$. Then, the platform's expected revenue under the full-disclosure policy is given by

$$\pi^{FD} := p\sum_{t=1}^{T} \left\{u_0 t \left(\mathbb{P}[\tau^+ = t] + \mathbb{P}[\tau^- = t]\right) + (T-t)\mathbb{P}[\tau^+ = t]\right\}. \tag{C.29}$$

Proof of Claim C.3. Let $\left(\tilde{\sigma}, \widetilde{\mathcal{H}}\right)$ denote the full-disclosure policy, whereby the message space $\widetilde{\mathcal{H}}$ consists of the platform's proprietary history $\widetilde{H}_t$. Define mapping $\varphi : \widetilde{\mathcal{H}} \to \mathcal{M} = \{1, 0, -1\}$ as

$$\varphi(\widetilde{H}_t) = \begin{cases} 1, & \text{if } \mathbb{E}[V|\widetilde{H}_t, \tilde{\sigma}] \geq v^{**}, \\ 0, & \text{if } \mathbb{E}[V|\widetilde{H}_t, \tilde{\sigma}] \in [v^*, v^{**}), \\ -1, & \text{if } \mathbb{E}[V|\widetilde{H}_t, \tilde{\sigma}] < v^*, \end{cases} \tag{C.30}$$

and denote $\gamma(\widetilde{H}_t) := \{(\varphi(\widetilde{H}_s), a_s) : s < t\}$. Further define the recommendation policy $(\sigma, \mathcal{M})$ to be

$$\sigma(m \mid H_t) = \frac{\sum_{\gamma(\widetilde{H}_t) = H_t, \varphi(\widetilde{H}_t) = m} \tilde{\sigma}\left(\widetilde{H}_t \,\middle|\, \widetilde{H}_t\right) \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right]}{\sum_{\gamma(\widetilde{H}_t) = H_t} \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right]} = \frac{\sum_{\gamma(\widetilde{H}_t) = H_t, \varphi(\widetilde{H}_t) = m} \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right]}{\sum_{\gamma(\widetilde{H}_t) = H_t} \mathbb{P}\left[\widetilde{H}_t \,\middle|\, \tilde{\sigma}\right]}, \tag{C.31}$$

for any $m \in \mathcal{M}$ and $H_t = \{(m_s, a_s) : m_s \in \mathcal{M}, a_s \in \{-1, 1\}, s < t\}$. Then, (C.30) immediately implies

$$(\sigma(m=1 \mid H_t), \sigma(m=0 \mid H_t), \sigma(m=-1 \mid H_t)) = \begin{cases} (1,0,0), \text{ if } \mathbb{E}[V|\gamma(\widetilde{H}_t) = H_t, \tilde{\sigma}] \geq v^{**}, \\ (0,1,0), \text{ if } \mathbb{E}[V|\gamma(\widetilde{H}_t) = H_t, \tilde{\sigma}] \in [v^*, v^{**}), \\ (0,0,1), \text{ if } \mathbb{E}[V|\gamma(\widetilde{H}_t) = H_t, \tilde{\sigma}] < v^*. \end{cases}$$

Following similar argument as in the proof of Proposition 2, we have full-disclosure policy $\tilde{\sigma}$ and σ induce the same purchase decisions from all customers and the same expected revenue for the platform. By (B.8) and (B.9), the above equation can also be rewritten as

$$(\sigma(m=1 \mid H_t), \sigma(m=0 \mid H_t), \sigma(m=-1 \mid H_t)) = \begin{cases} (1,0,0), \text{ if } \mathbb{E}[V|H_t, \sigma] \geq v^{**}, \\ (0,1,0), \text{ if } \mathbb{E}[V|H_t, \sigma] \in [v^*, v^{**}), \\ (0,0,1), \text{ if } \mathbb{E}[V|H_t, \sigma] < v^*, \end{cases}$$

which, by utilizing the definitions of n^-, n^+ and Proposition 3, can be expressed as

$$(\sigma(m=1 \mid H_t), \sigma(m=0 \mid H_t), \sigma(m=-1 \mid H_t)) = \begin{cases} (1,0,0), \text{ if } N(H_t) \geq n^+, \\ (0,1,0), \text{ if } N(H_t) \in (n^-, n^+), \\ (0,0,1), \text{ if } N(H_t) \leq n^-, \end{cases}$$

and hence, establishing the equivalence between recommendation policies σ and r given in (C.28).

By (4), customer t's purchase decision under the recommendation policy r given in (C.28) follows

$$a_t = \begin{cases} 1, & \text{if } N(H_t) \geq n^+, \\ S_t, & \text{if } N(H_t) \in (n^-, n^+), \\ -1, & \text{if } N(H_t) \leq n^-, \end{cases} \tag{C.32}$$

Thus, the net purchase position $N(H_t) \geq n^+$ (resp., $N(H_t) \leq n^-$) remains unchanged and generates a sale with probability 1 (resp., 0) after $t > \tau^+$ (resp., $t > \tau^-$). Then, utilizing (3), we can denote the expected revenue of a recommendation policy r given in (C.28) as follows

$$\begin{aligned} & p\sum_{t=1}^{T}\mathbb{P}\left[\tau^+ \wedge \tau^- = t\right]\left\{\mathbb{P}[a_t = 1 \mid m_t = 0, r]t + \mathbb{1}[\tau^+ = t](T-t)\right\} \\ &\quad = p\sum_{t=1}^{T}\{\left(\mathbb{P}[\tau^+ = t] + \mathbb{P}[\tau^- = t]\right) t \sum_n \underbrace{\mathbb{P}[S_t = 1 | N(H_t) = n]}_{= qv_n + (1-q)(1-v_n)} \underbrace{\mathbb{P}[N(H_t) = n | r]}_{z_t(n)} + \mathbb{P}[\tau^+ = t](T-t)\} \\ &\quad = p\sum_{t=1}^{T}\{\left(\mathbb{P}[\tau^+ = t] + \mathbb{P}[\tau^- = t]\right) t[q \underbrace{\sum_n v_n z_t(n)}_{= v_0 \text{ by (C.10)}} + (1-q) \underbrace{\sum_n v_n - v_n z_t(n))}_{= 1 - v_0 \text{ by (C.10) and (C.9)}}] + \mathbb{P}[\tau^+ = t](T-t)\}. \end{aligned}$$

Thus, (C.29) follows from $u_0 = qv_0 + (1-q)(1-v_0)$ definition given in Proposition 4. $\square$

Claim C.4. *The probabilities of positive and negative cascade occurring right after time $t \geq 1$ are given by*

$$\mathbb{P}\left[\tau^+ = t\right] = \begin{cases} [(1-u_0)(u_{-1})]^{\frac{t-1}{2}} u_0 \mathbb{1}[t \equiv 1 (mod 2)], & \text{for } v_0 \in [p, v^{**}), \\ (u_0)^{\frac{t}{2}}(1-u_1)^{\frac{t}{2}-1} u_1 \mathbb{1}[t \equiv 0 (mod 2)], & \text{for } v_0 \in [v^*, p), \end{cases} \quad \text{and} \tag{C.33}$$

$$\mathbb{P}\left[\tau^- = t\right] = \begin{cases} (1-u_0)^{\frac{t}{2}}(u_{-1})^{\frac{t}{2}-1}(1-u_{-1})\mathbb{1}[t \equiv 0 (mod 2)], & \text{for } v_0 \in [p, v^{**}), \\ [(u_0)(1-u_1)]^{\frac{t-1}{2}}(1-u_0)\mathbb{1}[t \equiv 1 (mod 2)], & \text{for } v_0 \in [v^*, p), \end{cases} \quad \text{respectively.} \tag{C.34}$$

Proof of Claim C.4. We now derive (C.33) and (C.34) for the case of $v_0 \in [p, v^{**})$. The case of $v_0 \in [v^*, p)$ can be derived in a similar fashion. For $v_0 \in [p, v^{**})$, in order for the net purchase position $N(H_t) = n$ to stay within (n^-, n^+) (i.e. $n \in (n^-, n^+)$), the first customer must not make the purchase (must receive a pessimistic signal, which occurs with probability $(1-u_0)$ as $N(H_1) = 0$) and the net purchase position gets updated to $N(H_2) = -1$; the second customer must make the purchase (must receive an optimistic signal, which occurs with probability u_{-1}) and the net purchase position gets updated to $N(H_3) = 0$; the third customer must not make the purchase (must receive a pessimistic signal, which occurs with probability $(1-u_0)$) and the net purchase position gets updated to $N(H_4) = -1$. By induction, the net purchase position must alternate between 0 and -1 in order for the net purchase position $N(H_t) = n$ to stay within (n^-, n^+), i.e. $N(H_t) = 0$ for odd t and $N(H_t) = -1$ for even t. Therefore, in order for τ^+ to be equal to t, a customer whose order of arrival is an odd number must make the purchase - must receive an optimistic signal (which occurs with probability u_0 as $N(H_t)$ would be 0). Similarly, in order for τ^- to be equal to t, a customer whose order of arrival is an even number must not make the purchase - must receive a pessimistic signal (which occurs with probability $1-u_{-1}$ as $N(H_t)$ would be -1). Hence, $\mathbb{P}[\tau^+ = t]$ and $\mathbb{P}[\tau^- = t]$ follows as in (C.33) and (C.34). $\square$

We now establish (C.27). We only demonstrate the case of $v_0 \in [p, v^{**})$ and odd T; all other cases follow similar argument. Re-writing (C.29) we have

$$\pi^{\text{FD}} = p\left[\sum_{t=1}^{T}[T + t(u_0 - 1)]\mathbb{P}[\tau^+ = t] + \sum_{t=1}^{T} u_0 t \mathbb{P}[\tau^- = t]\right]. \tag{C.35}$$

Utilizing (C.33) we can represent the first term of (C.35) as

$$=[T+(u_0-1)]u_0+[T+3(u_0-1)]u_0(1-u_0)(u_{-1})+\cdots+[T+(T-1)(u_0-1)]u_0(1-u_0)^{\frac{T-1}{2}}(u_{-1})^{\frac{T-1}{2}}.$$

Next, we obtain the following representation by multiplying the expression with $1-(1-u_0)u_{-1}$

$$\begin{aligned}
=&[T+(u_0-1)]u_0+[T+3(u_0-1)]u_0(1-u_0)(u_{-1})+\cdots+[T+(T-1)(u_0-1)]u_0(1-u_0)^{\frac{T-1}{2}}(u_{-1})^{\frac{T-1}{2}}\\
&-\Big[[T+(u_0-1)]u_0(1-u_0)(u_{-1})+[T+3(u_0-1)]u_0(1-u_0)^2(u_{-1})^2+\cdots+[T+(T-1)(u_0-1)]u_0(1-u_0)^{\frac{T+1}{2}}(u_{-1})^{\frac{T+1}{2}}\Big]\\
=&[T+(u_0-1)]u_0+2(u_0-1)u_0\Big[(1-u_0)(u_{-1})+(1-u_0)^2(u_{-1})^2+\cdots+(1-u_0)^{\frac{T-1}{2}}(u_{-1})^{\frac{T-1}{2}}\Big]\\
&-[T+(T-1)(u_0-1)]u_0(1-u_0)^{\frac{T+1}{2}}(u_{-1})^{\frac{T+1}{2}}\\
=&[T+(u_0-1)]u_0+2(u_0-1)u_0\Big[(1-u_0)(u_{-1})+(1-u_0)^2(u_{-1})^2+\cdots+(1-u_0)^{\frac{T-1}{2}}(u_{-1})^{\frac{T-1}{2}}+(1-u_0)^{\frac{T+1}{2}}(u_{-1})^{\frac{T+1}{2}}\Big]\\
&-[T+(T+1)(u_0-1)]u_0(1-u_0)^{\frac{T+1}{2}}(u_{-1})^{\frac{T+1}{2}}\\
=&[T+(u_0-1)]u_0+2(u_0-1)u_0(1-u_0)(u_{-1})\Big[1+(1-u_0)(u_{-1})+\cdots+(1-u_0)^{\frac{T-1}{2}}(u_{-1})^{\frac{T-1}{2}}\Big]\\
&-[T+(T+1)(u_0-1)]u_0(1-u_0)^{\frac{T+1}{2}}(u_{-1})^{\frac{T+1}{2}}\\
=&[T+(u_0-1)]u_0+\frac{2(u_0-1)u_0\Big[1-[(1-u_0)(u_{-1})]^{\frac{T+1}{2}}\Big]}{1-(1-u_0)u_{-1}}-[T+(T+1)(u_0-1)]u_0(1-u_0)^{\frac{T+1}{2}}(u_{-1})^{\frac{T+1}{2}},
\end{aligned}$$

where the last equality follows from the sum of geometric series formula. Then, we divide the expression back again by $1-(1-u_0)u_{-1}$ to get

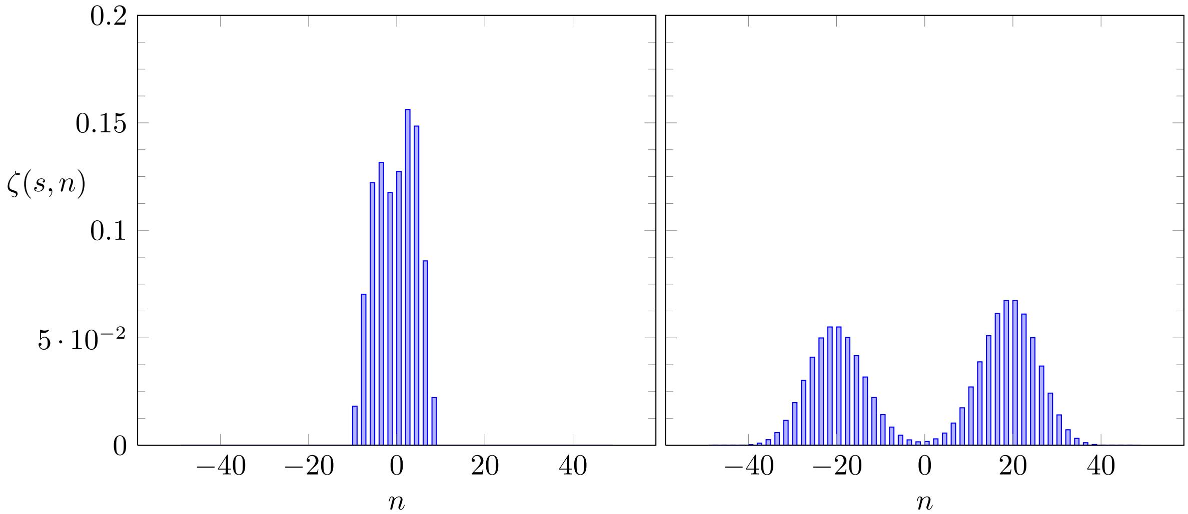
$$=\frac{[T+(u_0-1)]u_0-[T+(T+1)(u_0-1)]u_0(1-u_0)^{\frac{T+1}{2}}(u_{-1})^{\frac{T+1}{2}}}{1-(1-u_0)u_{-1}}+\frac{2(u_0-1)u_0\Big[1-[(1-u_0)(u_{-1})]^{\frac{T+1}{2}}\Big]}{[1-(1-u_0)u_{-1}]^2}.$$

By similar analysis, second term of (C.35) can be represented as

$$=-\frac{(T+1)u_0(1-u_{-1})(u_{-1})^{-1}[(1-u_0)u_{-1}]^{\frac{T+1}{2}}}{1-(1-u_0)u_{-1}}+\frac{2u_0(1-u_{-1})(1-u_0)\Big[1-[(1-u_0)(u_{-1})]^{\frac{T+1}{2}}\Big]}{[1-(1-u_0)u_{-1}]^2}.$$

Combining the above-derived representations of first and second term of (C.35) and a few simple steps of algebra, we reach at (C.27). ■

Appendix D: Additional Figures and Detailed Numerical Results



(a) $s = 10$. (b) $s = 50$.

Figure D.1 Distribution $\zeta(s,\cdot)$ (for $v_0 = .55$ and $p = q = .7$).

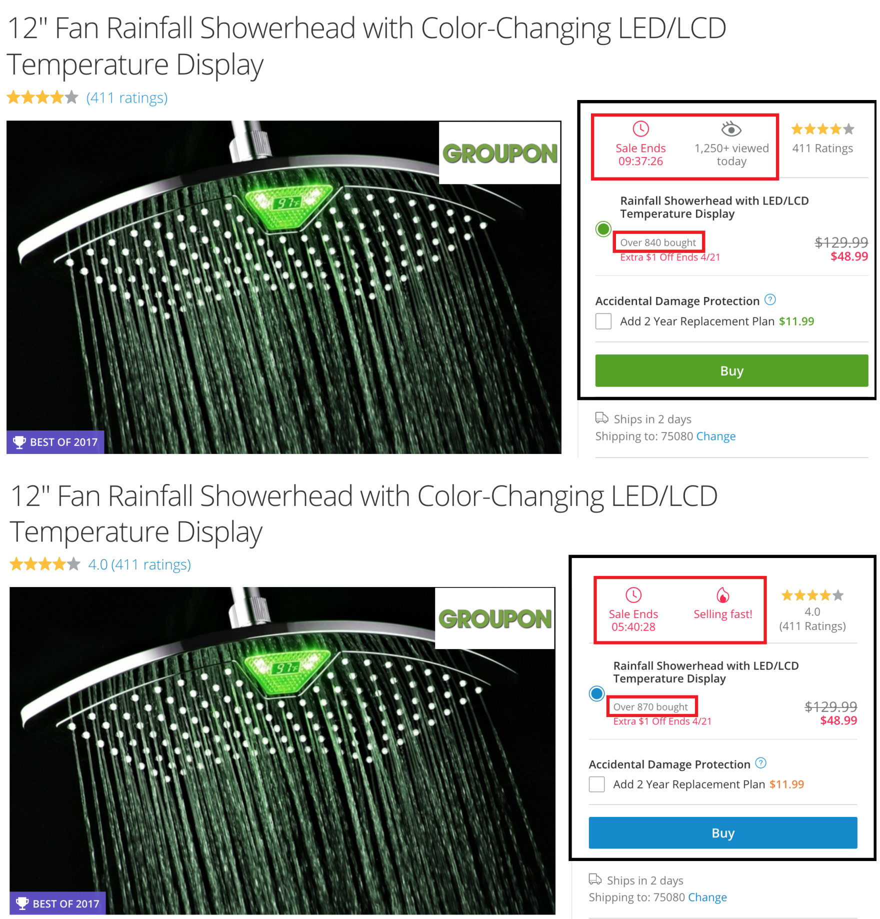


Figure D.2 Time-locked sales campaign example from Groupon.com.

Figure D.3 Time-locked sales campaign example from Amazon.com.

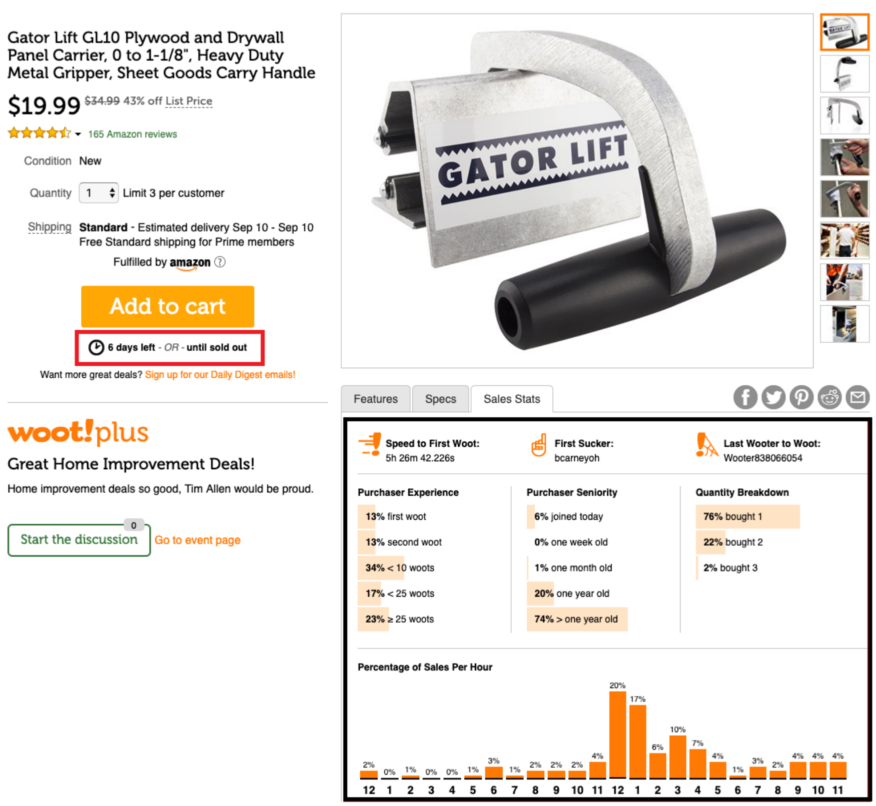


Figure D.4 Time-locked sales campaign example from Woot.com.

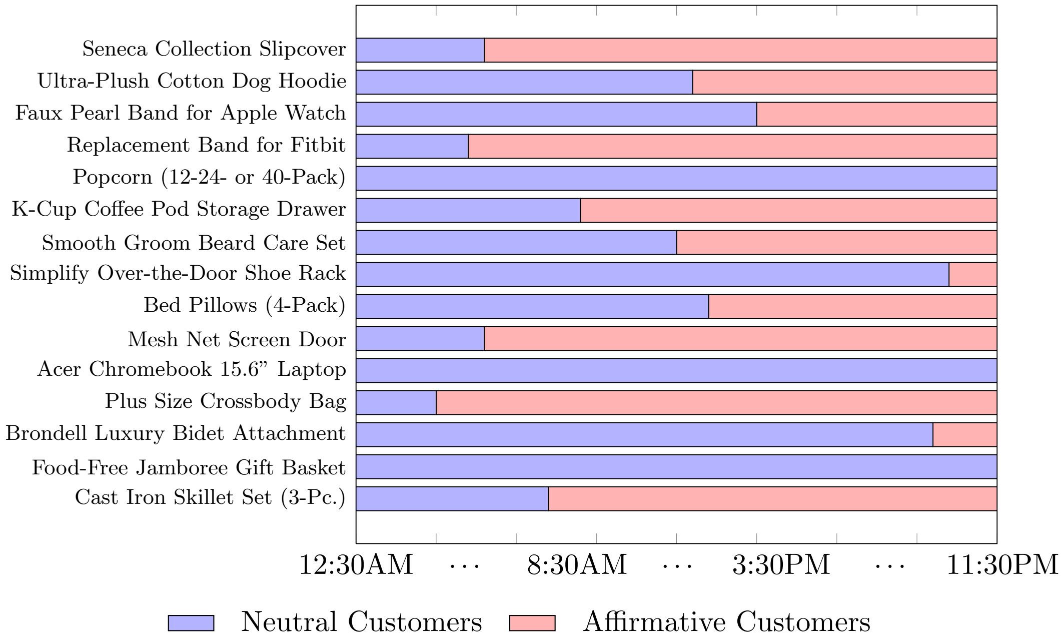


Figure D.5 Groupon daily deals on 4/8/19. Data collected in 30 minute intervals for 15 products. Groupon revealed no information or showed up-to-date count of visits to the product webpage during blue time stamps, and then switch to highlight the message "Selling Fast!" during red time stamps.

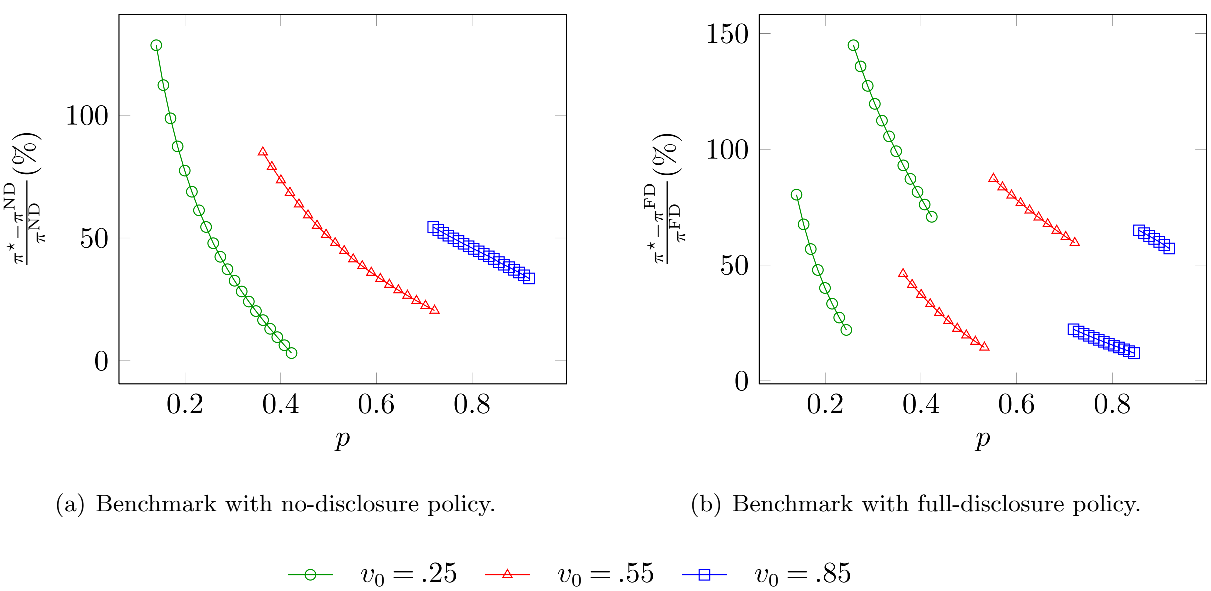


Figure D.6 Relative revenue performance of optimal policy $\pi^\star$ against no-disclosure policy π^{ND} and full-disclosure policy π^{FD} (for $q = .7$ and $T = 100$), plotted for the range of p such that $v_0 \in [v^*, v^{**})$ with the lower price limit corresponding to $v_0 = v^{**}$ and the upper price limit corresponding to $v_0 = v^*$.

Table D.1 $\pi^\star$ data used in Table 1.

$T = 50$

q	v_0	p = 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$q = 0.55$	0.05	0	0	0	0	0	0	0	0	0
	0.15	5	0	0	0	0	0	0	0	0
	0.25	5	10	0	0	0	0	0	0	0
	0.35	5	10	15	0	0	0	0	0	0
	0.45	5	10	15	20	16.155	0	0	0	0
	0.55	5	10	15	20	25	0	0	0	0
	0.65	5	10	15	20	25	30	0	0	0
	0.75	5	10	15	20	25	30	35	0	0
	0.85	5	10	15	20	25	30	35	40	0
	0.95	5	10	15	20	25	30	35	40	45
$q = 0.65$	0.05	0	0	0	0	0	0	0	0	0
	0.15	4.366	4.792	0	0	0	0	0	0	0
	0.25	5	7.842	8.297	0	0	0	0	0	0
	0.35	5	10	11.694	12.271	12.732	0	0	0	0
	0.45	5	10	15	15.995	16.653	17.133	0	0	0
	0.55	5	10	15	19.687	20.730	21.425	0	0	0
	0.65	5	10	15	20	25	25.919	26.573	0	0
	0.75	5	10	15	20	25	30	31.590	31.998	0
	0.85	5	10	15	20	25	30	35	37.839	37.304
	0.95	5	10	15	20	25	30	35	40	45
$q = 0.75$	0.05	1.575	0	0	0	0	0	0	0	0
	0.15	3.255	4.319	5.261	0	0	0	0	0	0
	0.25	5	6.330	7.554	8.625	9.375	0	0	0	0
	0.35	5	8.320	9.914	11.275	12.425	13.087	0	0	0
	0.45	5	10	12.234	14.012	15.475	16.613	17.750	0	0
	0.55	5	10	14.566	16.690	18.631	20.138	21.702	0	0
	0.65	5	10	15	19.382	21.712	23.788	25.656	27.545	0
	0.75	5	10	15	20	25	27.348	29.757	31.919	33.732
	0.85	5	10	15	20	25	30	33.766	36.486	38.734
	0.95	5	10	15	20	25	30	35	40	43.994
$q = 0.85$	0.05	1.158	1.910	0	0	0	0	0	0	0
	0.15	2.285	3.386	4.462	5.472	0	0	0	0	0
	0.25	3.408	4.894	6.311	7.702	8.999	10.133	0	0	0
	0.35	4.532	6.384	8.206	9.933	11.624	13.172	14.437	0	0
	0.45	5	7.876	10.069	12.221	14.248	16.212	17.936	19.059	0
	0.55	5	9.370	11.934	14.466	16.943	19.252	21.435	23.127	0
	0.65	5	10	13.802	16.712	19.582	22.375	24.934	27.194	29.565
	0.75	5	10	15	18.963	22.223	25.431	28.530	31.262	34.134
	0.85	5	10	15	20	25	28.491	32.046	35.440	38.707
	0.95	5	10	15	20	25	30	35	39.535	43.419
$q = 0.95$	0.05	0.582	1.040	1.494	1.940	0	0	0	0	0
	0.15	1.281	2.184	3.090	3.989	4.878	5.750	6.585	0	0
	0.25	1.978	3.340	4.686	6.038	7.382	8.709	10.005	11.222	0
	0.35	2.675	4.488	6.299	8.087	9.885	11.669	13.425	15.113	16.526
	0.45	3.372	5.637	7.900	10.158	12.388	14.628	16.845	19.004	20.931
	0.55	4.069	6.786	9.501	12.212	14.918	17.588	20.266	22.896	25.336
	0.65	4.767	7.935	11.102	14.266	17.426	20.579	23.686	26.787	29.740
	0.75	5	9.084	12.703	16.320	19.934	23.543	27.142	30.678	34.145
	0.85	5	10	14.304	18.374	22.443	26.508	30.567	34.610	38.550
	0.95	5	10	15	20	25	29.473	33.993	38.507	43.000

$T = 100$

q	v_0	p = 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$q = 0.55$	0.05	0	0	0	0	0	0	0	0	0
	0.15	10	0	0	0	0	0	0	0	0
	0.25	10	20	0	0	0	0	0	0	0
	0.35	10	20	30	0	0	0	0	0	0
	0.45	10	20	30	40	34.721	0	0	0	0
	0.55	10	20	30	40	50	0	0	0	0
	0.65	10	20	30	40	50	60	0	0	0
	0.75	10	20	30	40	50	60	70	0	0
	0.85	10	20	30	40	50	60	70	80	0
	0.95	10	20	30	40	50	60	70	80	90
$q = 0.65$	0.05	0	0	0	0	0	0	0	0	0
	0.15	8.784	9.619	0	0	0	0	0	0	0
	0.25	10	15.765	16.743	0	0	0	0	0	0
	0.35	10	20	23.533	24.873	26.042	0	0	0	0
	0.45	10	20	30	32.238	33.862	35.230	0	0	0
	0.55	10	20	30	39.571	41.833	43.681	0	0	0
	0.65	10	20	30	40	50	52.341	54.315	0	0
	0.75	10	20	30	40	50	60	63.799	65.650	0
	0.85	10	20	30	40	50	60	70	76.324	77.147
	0.95	10	20	30	40	50	60	70	80	90
$q = 0.75$	0.05	3.158	0	0	0	0	0	0	0	0
	0.15	6.547	8.682	10.537	0	0	0	0	0	0
	0.25	10	12.717	15.187	17.297	18.750	0	0	0	0
	0.35	10	16.731	19.905	22.666	24.925	26.189	0	0	0
	0.45	10	20	24.583	28.121	31.100	33.323	35.726	0	0
	0.55	10	20	29.272	33.518	37.381	40.458	43.677	0	0
	0.65	10	20	30	38.929	43.587	47.718	51.641	55.638	0
	0.75	10	20	30	40	50	54.887	59.749	64.382	68.576
	0.85	10	20	30	40	50	60	67.764	73.300	78.295
	0.95	10	20	30	40	50	60	70	80	88.314
$q = 0.85$	0.05	2.326	3.823	0	0	0	0	0	0	0
	0.15	4.593	6.807	8.950	10.959	0	0	0	0	0
	0.25	6.854	9.823	12.683	15.454	18.035	20.281	0	0	0
	0.35	9.118	12.822	16.460	19.950	23.320	26.400	28.899	0	0
	0.45	10	15.821	20.206	24.504	28.605	32.519	35.941	38.137	0
	0.55	10	18.824	23.953	29.014	33.961	38.638	42.983	46.325	0
	0.65	10	20	27.703	33.525	39.260	44.840	50.025	54.514	59.361
	0.75	10	20	30	38.041	44.562	50.974	57.164	62.702	68.527
	0.85	10	20	30	40	50	57.113	64.223	71.000	77.702
	0.95	10	20	30	40	50	60	70	79.216	87.004
$q = 0.95$	0.05	1.168	2.084	2.990	3.882	0	0	0	0	0
	0.15	2.570	4.382	6.193	7.990	9.767	11.507	13.174	0	0
	0.25	3.969	6.692	9.395	12.099	14.784	17.437	20.026	22.453	0
	0.35	5.368	8.994	12.614	16.208	19.802	23.368	26.878	30.247	33.057
	0.45	6.767	11.297	15.822	20.338	24.819	29.298	33.729	38.042	41.881
	0.55	8.166	13.600	19.029	24.451	29.862	35.228	40.581	45.837	50.705
	0.65	9.566	15.903	22.236	28.565	34.884	41.189	47.433	53.632	59.529
	0.75	10	18.206	25.444	32.678	39.906	47.124	54.320	61.427	68.353
	0.85	10	20	28.652	36.792	44.929	53.059	61.177	69.262	77.177
	0.95	10	20	30	40	50	58.995	68.034	77.062	86.047

$T = 500$

q	v_0	p = 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$q = 0.55$	0.05	0	0	0	0	0	0	0	0	0
	0.15	50	0	0	0	0	0	0	0	0
	0.25	50	100	0	0	0	0	0	0	0
	0.35	50	100	150	0	0	0	0	0	0
	0.45	50	100	150	200	193.144	0	0	0	0
	0.55	50	100	150	200	250	0	0	0	0
	0.65	50	100	150	200	250	300	0	0	0
	0.75	50	100	150	200	250	300	350	0	0
	0.85	50	100	150	200	250	300	350	400	0
	0.95	50	100	150	200	250	300	350	400	450
$q = 0.65$	0.05	0	0	0	0	0	0	0	0	0
	0.15	44.132	48.233	0	0	0	0	0	0	0
	0.25	50	79.151	84.405	0	0	0	0	0	0
	0.35	50	100	118.287	126.000	133.428	0	0	0	0
	0.45	50	100	150	162.339	172.116	181.460	0	0	0
	0.55	50	100	150	198.646	210.959	222.718	0	0	0
	0.65	50	100	150	200	250	264.180	277.773	0	0
	0.75	50	100	150	200	250	300	322.050	337.107	0
	0.85	50	100	150	200	250	300	350	384.738	400.006
	0.95	50	100	150	200	250	300	350	400	450
$q = 0.75$	0.05	15.825	0	0	0	0	0	0	0	0
	0.15	32.880	43.588	52.752	0	0	0	0	0	0
	0.25	50	63.811	76.259	86.672	0	0	0	0	0
	0.35	50	84.012	99.834	113.791	124.925	131.002	0	0	0
	0.45	50	100	123.368	140.996	156.100	167.011	179.662	0	0
	0.55	50	100	146.914	168.143	187.381	203.020	219.634	0	0
	0.65	50	100	150	195.304	218.587	239.155	259.607	280.827	0
	0.75	50	100	150	200	250	275.200	299.731	324.285	348.162
	0.85	50	100	150	200	250	300	339.760	367.928	395.429
	0.95	50	100	150	200	250	300	350	400	442.955
$q = 0.85$	0.05	11.669	19.124	0	0	0	0	0	0	0
	0.15	23.050	34.174	44.859	54.855	0	0	0	0	0
	0.25	34.426	49.256	63.652	77.472	90.321	101.468	0	0	0
	0.35	45.804	64.321	82.490	100.089	116.891	132.219	144.599	0	0
	0.45	50	79.387	101.297	122.764	143.462	162.970	179.984	190.762	0
	0.55	50	94.455	120.106	145.396	170.103	193.721	215.369	231.915	0
	0.65	50	100	138.918	168.029	196.689	224.556	250.754	273.067	297.901
	0.75	50	100	150	190.666	223.277	255.322	286.235	314.220	343.785
	0.85	50	100	150	200	250	286.094	321.637	355.482	389.682
	0.95	50	100	150	200	250	300	350	396.662	435.706
$q = 0.95$	0.05	5.860	10.436	14.963	19.416	0	0	0	0	0
	0.15	12.878	21.966	31.017	40.001	48.878	57.570	65.889	0	0
	0.25	19.895	33.508	47.070	60.585	74.007	87.264	100.192	112.300	0
	0.35	26.911	45.042	63.140	81.170	99.135	116.958	134.496	151.322	165.301
	0.45	33.928	56.577	79.198	101.776	124.263	146.652	168.800	190.345	209.481
	0.55	40.944	68.112	95.256	122.365	149.418	176.346	203.103	229.367	253.660
	0.65	47.961	79.646	111.314	142.954	174.551	206.071	237.407	268.390	297.840
	0.75	50	91.182	127.372	163.544	199.684	235.770	271.746	307.413	342.020
	0.85	50	100	143.431	184.134	224.818	265.469	306.054	346.476	386.199
	0.95	50	100	150	200	250	295.169	340.363	385.504	430.425

Table D.2 π^{NA} **data used in Table 1.**

$T = 50$

$q = 0.55$, v_0	p = 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.05	0	0	0	0	0	0	0	0	0
0.15	5	0	0	0	0	0	0	0	0
0.25	5	10	0	0	0	0	0	0	0
0.35	5	10	15	0	0	0	0	0	0
0.45	5	10	15	20	15.561	0	0	0	0
0.55	5	10	15	20	25	0	0	0	0
0.65	5	10	15	20	25	30	0	0	0
0.75	5	10	15	20	25	30	35	0	0
0.85	5	10	15	20	25	30	35	40	0
0.95	5	10	15	20	25	30	35	40	45

$q = 0.65$, v_0	p = 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.05	0	0	0	0	0	0	0	0	0
0.15	4.139	4.433	0	0	0	0	0	0	0
0.25	5	7.307	7.779	0	0	0	0	0	0
0.35	5	10	10.938	11.562	12.536	0	0	0	0
0.45	5	10	15	14.983	15.743	16.748	0	0	0
0.55	5	10	15	19.402	19.495	20.331	0	0	0
0.65	5	10	15	20	25	24.517	25.317	0	0
0.75	5	10	15	20	25	30	30.169	30.533	0
0.85	5	10	15	20	25	30	35	36.292	35.885
0.95	5	10	15	20	25	30	35	40	45

$q = 0.75$, v_0	p = 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.05	1.375	0	0	0	0	0	0	0	0
0.15	2.836	3.441	4.875	0	0	0	0	0	0
0.25	5	5.557	6.450	7.500	9.375	0	0	0	0
0.35	5	7.849	8.921	10.111	11.394	12.800	0	0	0
0.45	5	10	11.552	12.926	14.409	15.967	17.592	0	0
0.55	5	10	14.309	15.911	17.576	19.341	21.172	0	0
0.65	5	10	15	19.014	20.935	22.871	24.905	26.904	0
0.75	5	10	15	20	25	26.648	28.821	31.093	33.240
0.85	5	10	15	20	25	30	33.145	35.505	37.720
0.95	5	10	15	20	25	30	35	40	43.246

$q = 0.85$, v_0	p = 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.05	0.925	1.850	0	0	0	0	0	0	0
0.15	1.876	2.557	3.825	5.100	0	0	0	0	0
0.25	3.131	4.156	5.232	6.500	8.125	9.750	0	0	0
0.35	4.407	5.823	7.250	8.716	10.226	11.850	13.825	0	0
0.45	5	7.478	9.282	11.154	13.009	14.915	16.815	18.750	0
0.55	5	9.182	11.400	13.586	15.868	18.107	20.409	22.684	0
0.65	5	10	13.478	16.137	18.706	21.392	24.011	26.707	29.348
0.75	5	10	15	18.632	21.694	24.652	27.726	30.715	33.760
0.85	5	10	15	20	25	28.082	31.550	34.874	38.209
0.95	5	10	15	20	25	30	35	39.219	42.922

$q = 0.95$, v_0	p = 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.05	0.475	0.950	1.425	1.900	0	0	0	0	0
0.15	1.091	1.850	2.775	3.700	4.625	5.550	6.475	0	0
0.25	1.819	3.003	4.202	5.500	6.875	8.250	9.625	11.000	0
0.35	2.544	4.203	5.859	7.530	9.202	10.950	12.775	14.600	16.425
0.45	3.269	5.400	7.531	9.659	11.800	13.942	16.084	18.232	20.475
0.55	3.995	6.596	9.199	11.788	14.401	17.014	19.627	22.241	24.849
0.65	4.723	7.795	10.868	13.945	17.015	20.087	23.172	26.256	29.319
0.75	5	8.997	12.540	16.084	19.638	23.173	26.717	30.272	33.827
0.85	5	10	14.221	18.233	22.247	26.275	30.281	34.290	38.315
0.95	5	10	15	20	25	29.380	33.858	38.358	42.810

$T = 100$

$q = 0.55$, v_0	p = 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.05	0	0	0	0	0	0	0	0	0
0.15	10	0	0	0	0	0	0	0	0
0.25	10	20	0	0	0	0	0	0	0
0.35	10	20	30	0	0	0	0	0	0
0.45	10	20	30	40	33.732	0	0	0	0
0.55	10	20	30	40	50	0	0	0	0
0.65	10	20	30	40	50	60	0	0	0
0.75	10	20	30	40	50	60	70	0	0
0.85	10	20	30	40	50	60	70	80	0
0.95	10	20	30	40	50	60	70	80	90

$q = 0.65$, v_0	p = 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.05	0	0	0	0	0	0	0	0	0
0.15	8.466	9.094	0	0	0	0	0	0	0
0.25	10	15.030	16.066	0	0	0	0	0	0
0.35	10	20	22.554	23.952	25.793	0	0	0	0
0.45	10	20	30	30.977	32.691	34.769	0	0	0
0.55	10	20	30	39.183	40.307	42.264	0	0	0
0.65	10	20	30	40	50	50.604	52.631	0	0
0.75	10	20	30	40	50	60	61.825	63.737	0
0.85	10	20	30	40	50	60	70	74.364	75.375
0.95	10	20	30	40	50	60	70	80	90

$q = 0.75$, v_0	p = 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.05	2.750	0	0	0	0	0	0	0	0
0.15	5.813	6.930	9.750	0	0	0	0	0	0
0.25	10	11.335	13.088	15.000	18.750	0	0	0	0
0.35	10	15.965	18.141	20.544	23.032	25.630	0	0	0
0.45	10	20	23.386	26.282	29.296	32.388	35.550	0	0
0.55	10	20	28.885	32.191	35.726	39.344	43.030	0	0
0.65	10	20	30	38.416	42.346	46.477	50.684	54.937	0
0.75	10	20	30	40	50	53.970	58.539	63.313	67.940
0.85	10	20	30	40	50	60	67.015	72.169	77.103
0.95	10	20	30	40	50	60	70	80	87.416

$q = 0.85$, v_0	p = 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.05	1.850	3.700	0	0	0	0	0	0	0
0.15	3.810	5.116	7.650	10.200	0	0	0	0	0
0.25	6.358	8.413	10.521	13.000	16.250	19.500	0	0	0
0.35	8.916	11.765	14.648	17.559	20.511	23.700	27.650	0	0
0.45	10	15.137	18.780	22.514	26.225	29.986	33.748	37.540	0
0.55	10	18.523	23.015	27.467	32.013	36.519	41.091	45.637	0
0.65	10	20	27.213	32.533	37.788	43.144	48.441	53.822	59.146
0.75	10	20	30	37.571	43.696	49.747	55.908	61.989	68.071
0.85	10	20	30	40	50	56.529	63.449	70.311	77.160
0.95	10	20	30	40	50	60	70	78.860	86.395

$q = 0.95$, v_0	p = 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.05	0.950	1.900	2.850	3.800	0	0	0	0	0
0.15	2.196	3.700	5.550	7.400	9.250	11.100	12.950	0	0
0.25	3.657	6.029	8.412	11.000	13.750	16.500	19.250	22.000	0
0.35	5.117	8.430	11.752	15.080	18.412	21.900	25.550	29.200	32.850
0.45	6.579	10.840	15.094	19.366	23.638	27.914	32.188	36.468	40.950
0.55	8.043	13.242	18.459	23.654	28.870	34.086	39.306	44.519	49.742
0.65	9.511	15.650	21.807	27.970	34.104	40.265	46.425	52.589	58.737
0.75	10	18.067	25.156	32.264	39.372	46.446	53.550	60.654	67.758
0.85	10	20	28.524	36.559	44.613	52.667	60.681	68.726	76.772
0.95	10	20	30	40	50	58.856	67.855	76.854	85.801

$T = 500$

$q = 0.55$, v_0	p = 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.05	0	0	0	0	0	0	0	0	0
0.15	50	0	0	0	0	0	0	0	0
0.25	50	100	0	0	0	0	0	0	0
0.35	50	100	150	0	0	0	0	0	0
0.45	50	100	150	200	190.209	0	0	0	0
0.55	50	100	150	200	250	0	0	0	0
0.65	50	100	150	200	250	300	0	0	0
0.75	50	100	150	200	250	300	350	0	0
0.85	50	100	150	200	250	300	350	400	0
0.95	50	100	150	200	250	300	350	400	450

$q = 0.65$, v_0	p = 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.05	0	0	0	0	0	0	0	0	0
0.15	43.329	46.799	0	0	0	0	0	0	0
0.25	50	77.688	83.414	0	0	0	0	0	0
0.35	50	100	116.670	124.568	133.046	0	0	0	0
0.45	50	100	150	160.298	170.247	180.782	0	0	0
0.55	50	100	150	197.848	208.496	220.440	0	0	0
0.65	50	100	150	200	250	261.284	275.130	0	0
0.75	50	100	150	200	250	300	318.710	334.151	0
0.85	50	100	150	200	250	300	350	381.282	397.157
0.95	50	100	150	200	250	300	350	400	450

$q = 0.75$, v_0	p = 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.05	13.750	0	0	0	0	0	0	0	0
0.15	29.738	34.913	48.750	0	0	0	0	0	0
0.25	50	57.892	66.353	75.000	93.750	0	0	0	0
0.35	50	81.072	92.685	104.407	116.287	128.291	0	0	0
0.45	50	100	119.198	134.107	149.083	164.220	179.431	0	0
0.55	50	100	146.034	163.979	182.160	200.377	218.766	0	0
0.65	50	100	150	194.187	215.415	236.846	258.284	279.907	0
0.75	50	100	150	200	250	273.535	298.177	322.794	347.457
0.85	50	100	150	200	250	300	338.698	366.206	393.875
0.95	50	100	150	200	250	300	350	400	441.374

$q = 0.85$, v_0	p = 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.05	9.250	18.500	0	0	0	0	0	0	0
0.15	19.325	25.586	38.250	51.000	0	0	0	0	0
0.25	32.210	42.509	52.863	65.000	81.250	97.500	0	0	0
0.35	45.105	59.449	73.906	88.359	102.848	118.500	138.250	0	0
0.45	50	76.491	94.960	113.504	132.069	150.641	169.258	187.888	0
0.55	50	93.511	116.125	138.691	161.325	183.992	206.644	229.362	0
0.65	50	100	137.241	163.984	190.641	217.384	244.129	270.854	297.603
0.75	50	100	150	189.231	220.074	250.815	281.689	312.477	343.268
0.85	50	100	150	200	250	284.417	319.226	354.185	389.033
0.95	50	100	150	200	250	300	350	396.137	434.943

$q = 0.95$, v_0	p = 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.05	4.750	9.500	14.250	19.000	0	0	0	0	0
0.15	11.035	18.500	27.750	37.000	46.250	55.500	64.750	0	0
0.25	18.391	30.235	42.096	55.000	68.750	82.500	96.250	110.000	0
0.35	25.742	42.313	58.904	75.499	92.094	109.500	127.750	146.000	164.250
0.45	33.093	54.413	75.713	97.042	118.371	139.700	161.027	182.363	204.750
0.55	40.444	66.499	92.555	118.585	144.649	170.712	196.776	222.825	248.894
0.65	47.814	78.586	109.376	140.166	170.926	201.724	232.522	263.319	294.091
0.75	50	90.675	126.199	161.723	197.248	232.737	268.268	303.800	339.300
0.85	50	100	143.026	183.284	223.542	263.800	304.017	344.281	384.546
0.95	50	100	150	200	250	294.845	339.832	384.820	429.765

Table D.3 Data used in Figure D.6.

$q = 0.7, T = 100$		$v_0 = 0.25$		
		π^*	π^{ND}	π^{FD}
	0.140	12.785	5.595	7.087
	0.155	13.139	6.190	7.841
	0.170	13.486	6.786	8.595
	0.185	13.824	7.381	9.349
	0.199	14.153	7.976	10.103
	0.214	14.473	8.571	10.857
	0.229	14.783	9.167	11.611
	0.244	15.081	9.762	12.365
	0.259	15.316	10.357	6.253
p	0.274	15.591	10.952	6.613
	0.289	15.853	11.548	6.972
	0.304	16.101	12.143	7.331
	0.318	16.333	12.738	7.691
	0.333	16.549	13.333	8.050
	0.348	16.748	13.929	8.410
	0.363	16.928	14.524	8.769
	0.378	17.088	15.119	9.128
	0.393	17.228	15.714	9.488
	0.408	17.344	16.310	9.847
	0.423	17.435	16.905	10.207

$q = 0.7, T = 100$		$v_0 = 0.55$		
		π^*	π^{ND}	π^{FD}
	0.363	34.869	18.857	23.857
	0.382	35.511	19.839	25.100
	0.400	36.133	20.821	26.343
	0.419	36.733	21.804	27.585
	0.438	37.309	22.786	28.828
	0.457	37.857	23.768	30.070
	0.476	38.378	24.750	31.313
	0.495	38.954	25.732	32.556
	0.514	39.527	26.714	33.798
p	0.533	40.094	27.696	35.041
	0.552	40.543	28.679	21.645
	0.570	41.102	29.661	22.387
	0.589	41.651	30.643	23.128
	0.608	42.192	31.625	23.869
	0.627	42.728	32.607	24.611
	0.646	43.250	33.589	25.352
	0.665	43.759	34.571	26.093
	0.684	44.249	35.554	26.834
	0.703	44.731	36.536	27.576
	0.721	45.199	37.518	28.317

$q = 0.7, T = 100$		$v_0 = 0.85$		
		π^*	π^{ND}	π^{FD}
	0.719	71.064	46.008	58.138
	0.729	71.548	46.683	58.990
	0.740	72.029	47.357	59.843
	0.750	72.507	48.032	60.695
	0.761	72.982	48.706	61.548
	0.772	73.452	49.381	62.400
	0.782	73.920	50.056	63.253
	0.793	74.383	50.730	64.105
	0.803	74.837	51.405	64.958
p	0.814	75.283	52.079	65.810
	0.824	75.719	52.754	66.663
	0.835	76.143	53.429	67.515
	0.845	76.554	54.103	68.367
	0.856	76.779	54.778	46.545
	0.866	77.155	55.452	47.118
	0.877	77.517	56.127	47.691
	0.888	77.851	56.802	48.264
	0.898	78.144	57.476	48.838
	0.909	78.383	58.151	49.411
	0.919	78.547	58.825	49.984